Stochastic dynamics of reaction-diffusion systems: An epidemic model case study

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Reaction-diffusion systems are ubiquitous

• Extensively used in modelling dynamical processes in physics, chemistry, biology, ecology, etc

• Typical representations:
  \[
  \frac{d\vec{\phi}}{dt} = \vec{F}(\vec{\phi}) + \mathbf{D} \cdot \nabla^2 \vec{\phi}
  \]

• Problem: Neglect of inevitable intrinsic noises:
  – Reaction & diffusion fluctuations

• Goal: To understand the importance of these fluctuations, and how to incorporate them into the analysis
SIRS epidemic model as a case study

• A infectious disease model with 3 types of individuals
  – $S$: density of susceptible individuals
  – $I$: density of infected individuals
  – $R$: density of recovered individuals

Infection:

rate: $\alpha$
SIRS epidemic model as a case study

• A infectious disease model with 3 types of individuals
  – S : density of susceptible individuals
  – I : density of infected individuals
  – R : density of recovered individuals

Recovery: \( I \) \( \rightarrow \) \( R \)

rate: \( \beta \)
SIRS epidemic model as a case study

- A infectious disease model with 3 types of individuals
  - $S$: density of susceptible individuals
  - $I$: density of infected individuals
  - $R$: density of recovered individuals

Re-infection possible: $\gamma$

rate: $\gamma$
Epidemic evolution

\[
\begin{align*}
\frac{dS}{dt} &= -\alpha SI + \gamma R \\
\frac{dI}{dt} &= \alpha SI - \beta R \\
\frac{dR}{dt} &= \beta I - \gamma R
\end{align*}
\]
Steady-state solution

• 1 conservation law: \[ N = S(t) + I(t) + R(t) \]

• 1 stable fixed point: \[ [N = 1, \beta = 2, \gamma = 1] \]
How about diffusion?

• Typical setup:
  \[ \begin{align*}
  \partial_t S &= D \nabla^2 S - \alpha SI + \gamma R \\
  \partial_t I &= D \nabla^2 I + \alpha SI - \beta R \\
  \partial_t R &= D \nabla^2 R + \beta I - \gamma R
  \end{align*} \]

• Linear stability from mean-field fixed point at \( \alpha_{\text{crit}} \), i.e., consider
  \[ S(x, t) = 1 + \epsilon_S e^{\sigma t + ik \cdot x} \], etc

• Find:
  \[ \sigma = -Dk^2 + (\alpha - \beta) \quad \rightarrow \text{diffusion has no effect on stability} \]
Phase diagram from linear stability analysis

\[ N = 0.1 \text{ agent per lattice site, } \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1} \]

\[ \alpha_{\text{crit}} \]

Active phase

Passive phase

Is this true?
What are we actually modelling?
A microscopic picture

- $S$, $R$, $I$ hop to the neighbouring site with rate $\Gamma$

$$D = \Gamma \Delta x^2$$

- $S + I \rightarrow I + I$
happens with rate $\alpha$ when $S$ and $I$ overlap
Phase diagram from linear stability analysis

\[ N = 0.1 \text{ agent per lattice site, } \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1} \]

Active phase

Passive phase

\[ \alpha_{\text{crit}} \]

Mean-field result
Lattice simulation vs. MF result

$[N = 0.1 \text{ agent per lattice site}, \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1}]$

What is missing?
Neglect of fluctuations in reaction and diffusion events!
We can do better!

\[ N = 0.1 \text{ agent per lattice site, } \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1} \]

- I will now present a theory that incorporates all the fluctuations into the analysis
- The theory is called the Doi-Peliti formalism and is based on quantum field theory
But first, a bit of history on fluctuation analysis

• Forget about diffusion, and focus on the simple reaction system

\[ A + A \xrightarrow{\lambda} \emptyset \]

• Macroscopic equation:

\[ \frac{dn}{dt} = -2\lambda n^2 \]

• To capture fluctuations – Chemical Master Equation (CME):

\[ \frac{d}{dt}P(n, t) = \lambda [(n + 2)(n + 1)P(n + 2, t) - n(n - 1)P(n, t)] \]

• Unfortunately, the CME is usually intractable
  \[ \rightarrow \text{need approximations} \]
Kramers-Moyal (KM) expansion

- Start from CME:
  \[ \frac{d}{dt} P(n, t) = \sum_{h} [W(n, h)P(h, t) - W(h, n)P(n, t)] \]

- Assume transitions only happen in neighbouring states:

  \[
  \frac{\partial}{\partial t} P(n, t) = \int dh [W(n, n-h)P(n-h, t) - W(h, n)P(n, t)] \\
  = \int dh \sum_{l=1}^{\infty} \frac{(-h)^l}{l!} \frac{\partial^l}{\partial n^l} [\eta_l(n)P(n, t)]
  \]

  where \( \eta_l(n) = \int dh (n-h)^l W(h, n) \)

- For the process: \( A + A \xrightarrow{\lambda} \emptyset \), we have

  \[
  \frac{\partial}{\partial t} P(n, t) = 2\lambda \frac{\partial}{\partial n} [n^2 P(n, t)] + 2\lambda \frac{\partial^2}{\partial n^2} [n^2 P(n, t)]
  \]

Drift \quad \text{Fluctuations}
van Kampen approximation

- Also known as system size expansion, $\Omega$ expansion, or linear noise approximation

- Define new variables:  \[ n(t) = \Omega \phi(t) + \sqrt{\Omega} x(t) \]

- Substitute back into KM expansion to get two equations:
  - One for the intensive variable and one for the fluctuations

\[
\frac{d\phi}{dt} = -2\lambda \phi^2 \\
\frac{\partial}{\partial t} P(x, t) = 4\lambda \phi(t) \frac{\partial}{\partial x} [xP(x, t)] + \lambda \phi(t)^2 \frac{\partial^2}{\partial x^2} P(x, t)
\]
Problems

• These approximations fail if the system size $\Omega$ (i.e., volume, total particle numbers, etc) is small

• Population can become negative due to fluctuations – boundary conditions not captured properly

• **Solutions**: keep track of the boundary condition and the discrete nature of problem seriously by using raising and lowering operators
Operator formalism

• Start with the empty state, the state with no particles: $|0\rangle$

• Define raising operator: $a^\dagger |n\rangle = |n + 1\rangle$
  and lowering operator: $a|n\rangle = n|n - 1\rangle$

• In vector and matrix form:

  $|0\rangle = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} \vdots \\ 1 \\ 0 \end{pmatrix} \quad a^\dagger = \begin{pmatrix} \ddots & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \quad a = \begin{pmatrix} \ddots \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

• In particular: $a^\dagger a |n\rangle = n|n\rangle \quad [a, a^\dagger] \equiv aa^\dagger - a^\dagger a = 1$
Operator representation of CME

• Back to \( A + A \xrightarrow{\lambda} \emptyset \), with CME:

\[
\frac{d}{dt} P(n, t) = \lambda [(n + 2)(n + 1)P(n + 2, t) - n(n - 1)P(n, t)]
\]

• To use operators, first define: \( |\psi(t)\rangle \equiv \sum_n P(n, t)|n\rangle \)

• Then

\[
\frac{d}{dt} |\psi(t)\rangle = \lambda(1 - a^\dagger a) a^2 \sum_n P(n, t)|n\rangle
\]

\[= -\hat{H} |\psi(t)\rangle \quad \text{Schrödinger equation}
\]

• Formal solution: \( |\psi(t)\rangle = \exp(-\hat{H}t)|\psi(0)\rangle \)

Erwin Schrödinger, Nobel Prize ‘33
Getting rid of non-commutative operators

• Introduce coherent states:

\[ |\phi\rangle = \exp \left( -\frac{1}{2} |\phi|^2 + \phi a^\dagger \right) |0\rangle \]

where \( \phi \) is a complex number

• Slice up the temporal evolution:

\[ e^{-\hat{H}t} = \int \ldots |\phi_{t+\Delta t}\rangle \langle \phi_{t+\Delta t}| e^{-\hat{H}\Delta t} |\phi_t\rangle \langle \phi_t| e^{-\hat{H}\Delta t} |\phi_{t-\Delta t}\rangle \langle \phi_{t-\Delta t}| \ldots \]

Path integral method

Roy Glauber
Nobel Prize ‘05

Richard Feynman
Nobel Prize ‘65
Field theory—it’s all about integration

• Do a lot of integration to compute average density

\[ \bar{n}(t) = \int \mathcal{D}\phi \mathcal{D}\phi \phi(t) e^{-S} \]

where

\[ S = \int d^d x dt \left[ \bar{\phi}(\partial_t - D \nabla^2)\phi + 2\lambda \bar{\phi}\phi^2 + \lambda \bar{\phi}^2 \phi^2 - n_0 \bar{\phi} \delta(t) \right] \]

\[ \mathcal{D}\bar{\phi} \mathcal{D}\phi = \prod_k \left( \frac{d(\text{Im}\phi_{tk}) d(\text{Re}\phi_{tk})}{\pi} \right) \]
Doi-Peliti vs. Kramers-Moyal

• In terms of Langevin’s equations:

Kramers-Moyal: \[ d\bar{n} = -2\lambda \bar{n}^2 dt + \sqrt{\lambda \bar{n}} dw \] \[ \times \]

Doi-Peliti: \[ d\bar{n} = -2\lambda \bar{n}^2 dt + i\sqrt{\lambda \bar{n}} dw \] \[ \checkmark \]
Nothing’s perfect ...

• **Problem:** we can’t really do the integration!

\[
\bar{n}(t) = \int \mathcal{D}\bar{\phi} \mathcal{D}\phi \phi(t) e^{-S}
\]

• **But:** Doi-Peliti formalism enables us to do the integration partially based on physical pictures
So what did I calculate?

- Specifically, I considered the following interactions in the calculation:

[For the experts, I sum over all the sequential one-loop diagrams]
Agreements with simulation results

\[ N = 0.1 \text{ agent per lattice site}, \beta = \gamma = 5 \times 10^{-4} s^{-1} \]
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Conclusion

• Mobility and fluctuations have critical effects on reaction-diffusion systems

• Doi-Peliti field-theoretical formalism is a method to analyse these effects

• Need further method development to provide a more comprehensive treatment of the problem
Thanks

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References

- **Today’s talk:**

- **References on the Doi-Peliti method:**

Thank you!