# Dissipativity characterization of a class of oscillators and networks of oscillators

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***Abstract*— This paper uses dissipativity theory to provide the system-theoretic description of a basic oscillation mechanism. Elementary input-output tools are then used to prove the existence and stability of limit cycles in these "oscillators".**

**The main benefit of the proposed approach is that it is well suited for the analysis and design of interconnections, thus providing a valuable mathematical tool for the study of networks of coupled oscillators.**

**Keywords:** limit cycles; dissipativity; HOPF bifurcation; networks of oscillators.

## I. INTRODUCTION

Oscillations in physical systems result from a sustained energy exchange between two or several storage elements. A basic mechanism for orchestrating the energy exchange is through the presence of a (static) element that delivers energy to the system when its energy is low and dissipates its energy when it is high. The VAN DER POL oscillator is the simplest electrical realization of this mechanism, the energy exchange between an inductor and a capacitor being regulated by an active element, modeled as a static resistance with a negative characteristic at low energy and with a positive characteristic at high energy.

The aim of the present paper is to characterize a class of "oscillators" that fits this energy description, and to study the existence and stability of limit cycles in such systems by relying on their dissipativity properties.

An obvious benefit of this input-output approach for the characterization of limit cycles is that it is not restricted to low-dimensional systems. This advantage has made for instance the describing function method a popular tool to study limit cycles, even though this method is only approximate. A further benefit of the dissipativity approach is that it is well-suited for the analysis of interconnections. A central motivation for this paper is to show that the characterization of a stable limit cycle for one oscillator extends in a straightforward manner when several such oscillators are arranged in a network configuration through symmetric linear coupling.

The study of networks of coupled oscillators has been an active research area over the last decade in biology and physics [2], [3], [4], [8], [10]. A dominant tool for the mathematical analysis of limit cycles in such systems is the description of each isolated oscillator by a single phase variable [6], [13], [14]. Making this reduction procedure rigorous usually requires weak coupling between the oscillators, an assumption not made in the present paper.

The paper is organized as follows. In Section II, we introduce a dissipativity-based mechanism for oscillations. In Section III, we characterize sufficient conditions for the existence of a stable limit cycle for "dissipative oscillators". In Section IV, the result is extended to networks of such oscillators, assuming linear coupling. A numerical illustration is provided in Section V.

Our approach is based on passivity and related concepts. Passivity is strongly linked to the stability property of the system. In fact, under detectability conditions, passivity implies LYAPUNOV stability [11]. For the mathematical definitions of passivity and other related concepts the reader is referred to [11], [15] and [7].

## II. A DISSIPATIVITY MECHANISM FOR OSCILLATIONS

The block diagram of Figure 1 illustrates the architecture of the "oscillators" considered in this paper. The oscillations result from the energy exchange between a linear time-invariant (LTI) passive system $H$ with state $\mathbf{x}_H$ and storage function $S_a(\mathbf{x}_H)$ and a (lossless) pure integrator with state $\xi$. We denote by $P(s)$ the transfer function associated to $H$.

Their feedback interconnection is a passive system with storage function $S_a(\mathbf{x}_H) + \frac{1}{2}\xi^2$. The sustained oscillation will be created by the static nonlinearity $\psi(\sigma, y) = -k_p y + \phi(\sigma)y$ which, when put in feedback with the linear system, yields a nonlinear system $\Pi$ characterized by the dissipation inequality

$$\dot{S} \le k_p y^2 - \phi(\sigma)y^2 \qquad (1)$$

where $S$ is the storage function of the nonlinear system $\Pi$.

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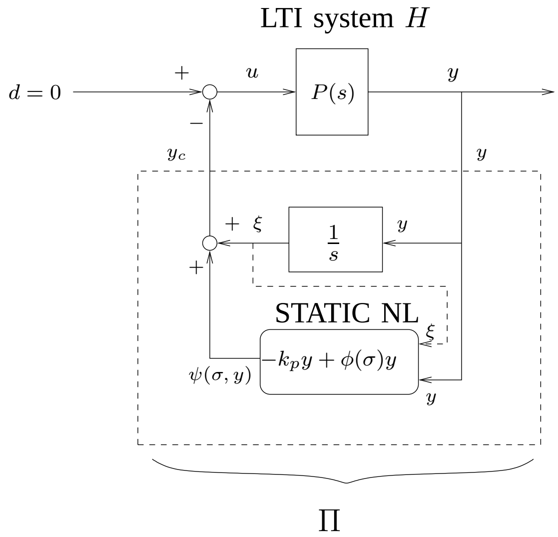


Fig. 1. A block diagram of the dissipative oscillator.

The nonlinearity $\phi(.)$ is a smooth, positive definite and radially unbounded function such that $\phi(0) = \phi'(0) = 0$ and $\phi''(0) \neq 0$. An example of such a nonlinearity is $\phi(.) = (.)^2$. This nonlinearity makes the total storage $S$ decrease when $\sigma$ is sufficiently large. Its argument $\sigma$ is left deliberately general but must provide some (time-invariant) measure of the energy stored in the system.

The parameter $k_p > 0$ models the "active" part of the static nonlinearity creating a constant positive output feedback in the overall system.

The fact that this positive feedback is counteracted by the nonlinearity $\phi$ at high energy and not counteracted at low energy provides the oscillation mechanism.

The feedback system just described reduces to two well-known oscillators when $H$ is the pure integrator $\dot{y} = u$ : the VAN DER POL oscillator is obtained for $\phi(\sigma) = \xi^2$ and the RAYLEIGH oscillator is obtained for $\phi(\sigma) = y^2$. These two second-order systems are well known in the litterature. They possess a unique equilibrium at the origin $(y, \xi) = (0, 0)$. This equilibrium is globally asymptotically stable for $k_p \leq 0$ and undergoes a supercritical HOPF bifurcation at $k_p = 0$. Moreover, the resulting limit cycle is unique and globally asymptotically stable.

## III. EXISTENCE AND STABILITY ANALYSIS OF A LIMIT CYCLE IN THE DISSIPATIVE OSCILLATOR

If the dissipative oscillator described in Section II is a correct higher-dimensional generalization of the second-order VAN DER POL and RAYLEIGH oscillators obtained for $P(s) = \frac{1}{s}$, a globally asymptotically stable limit cycle is expected to arise from a supercritical HOPF bifurcation at a critical value of the activation parameter $k_p$. Based on the HOPF bifurcation theorem, the next result provides sufficient conditions for the existence and local stability of the limit cycle.

*Theorem 3.1:* Consider the feedback interconnection $\Pi$ of Figure 1. Denote by $M$ the Jacobian matrix associated with the linearization of $\Pi$ at the origin and by $P(s)$ the transfer function associated with the LTI system $H$. Assume that $H$ is passive, controllable, detectable, of relative degree one, and such that $P(0) \neq 0$. If $\phi(.)$ is a smooth, positive definite, and radially unbounded function such that $\phi(0) = \phi'(0) = 0$ and $\phi''(0) \neq 0$, then,

**1)** There exists a critical value $k_p^* \geq 0$ of the parameter $k_p$ such that all the eigenvalues of $M$ have non positive real parts for $k_p \leq k_p^*$ and that two nonzero complex conjugate eigenvalues of $M$ cross the imaginary axis at $k_p = k_p^*$.

**2)** If no more than two simple eigenvalues of $M$ are on the imaginary axis at $k_p = k_p^*$, then the system $\Pi$ undergoes a HOPF bifurcation at $k_p = k_p^*$. If $H$ is OFP($k_p^*$) [1] then this bifurcation is supercritical and gives rise to an asymptotically stable limit cycle for $k_p \gtrsim k_p^*$. [2]

*Proof:*

**Part 1)**

At equilibrium, $\dot{\xi}(t) = 0 = y(t) = -P(0)\xi$. This implies that $\xi = 0$ since $P(0) \neq 0$. From the detectability assumption of $H$, we conclude that $\mathbf{x}_H \to 0$ as $t \to \infty$. $\Pi$ is thus detectable and has the origin as unique equilibrium point.

Let $R(s)$ denote the transfer function of the feedback system $\Pi$ linearized around the origin. Since $H$ is controllable, the eigenvalues of the linearization $M$ are the unobservable modes of $H$, necessarily asymptotically stable since $H$ is detectable, and the poles of $R(s)$.

Let $Q(s) = \frac{sP(s)}{s+P(s)}$. Since $R(s) = \frac{Q(s)}{1-k_pQ(s)}$, the position of the poles of $R(s)$ as a function of the parameter $k_p \geq 0$ is given by the (negative) root locus of $Q(s)$ in the complex plane.

The root locus starts at the poles of $Q(s)$ for $k_p = 0$. Since $Q(s)$ is the feedback interconnection of two passive systems it is passive and all its poles are in the closed left-half plane. As a consequence, the root locus starts in the closed left-half plane. The root locus ends at the zeros of $Q(s)$. The finite zeros of $Q(s)$ are those of $P(s)$, all in the closed left-half plane, plus a zero at the origin.

The intersections of the negative root locus with the real axis are given by the parts of the real axis located at the right side of an odd number of singularities (pole(s) or zero(s)). Because the total number of poles and zeros is odd and because they are all located in the closed left-half plane, the entire positive real axis is part of the root locus. By continuity, two distinct branches must cross the imaginary axis at some critical value $k_p^*$.

**Part 2)**

At $k_p = k_p^*$, the system is passive and possesses a center manifold of dimension two. Using the KALMAN-

[1] $H$ is output feedback passive (OFP) if it is dissipative with respect to $w(u, y) = uy - \rho y^2, \rho \in \mathbb{R}$, i.e. $\exists S(\mathbf{x}) \geq 0$ s.t. $\dot{S} \leq uy - \rho y^2$ [11].

[2] $A \gtrsim B$ means $A$ greater than $B$ but sufficiently close to $B$.

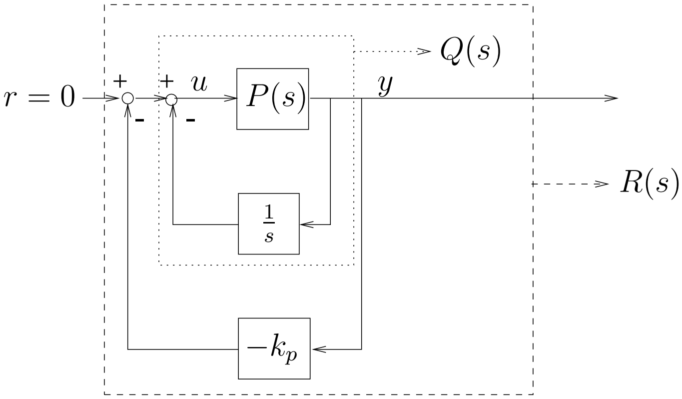


Fig. 2. $R(s)$ expressed as the feedback interconnection of $Q(s)$ and proportional gain $k_p$.

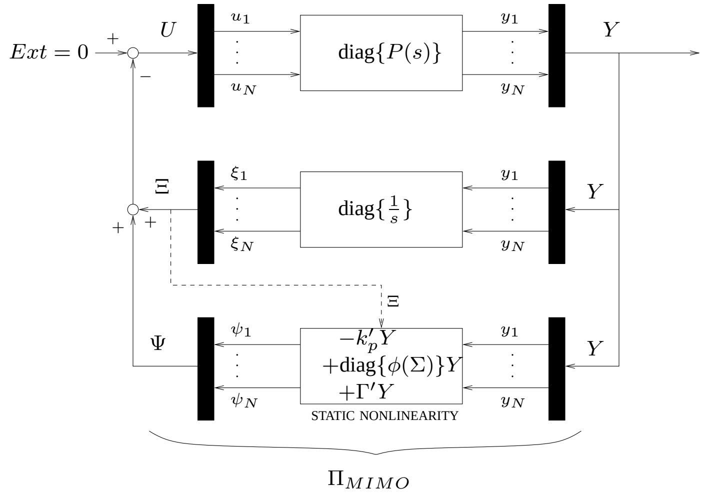


Fig. 3. MIMO representation of a network of interconnected oscillators. diag$\{\phi(\Sigma)\}$ =diag$\{(\phi(\sigma_1), \ldots, \phi(\sigma_N))\}$.

YAKUBOVICH-POPOV lemma [11], the reduced dynamics are of the form $\dot{x} = Jx - (d^Tx)^2bb^Tx + \mathcal{O}(||x||^4),\ x \in \mathbb{R}^2$ with $J = -J^T$ and $b \neq 0,\ d \neq 0$. This system has a LYAPUNOV function that verifies $\dot{V} \leq ||x||^4 + \mathcal{O}(||x||^5)$, which proves 3-asymptotic stability (3-as) of $x = 0$, a sufficient condition for the HOPF bifurcation to be supercritical [1, Theorem 7.2.3].

■

*Remark 3.2:* Theorem 3.1 is merely a local result and can be alternatively proven by applying a frequency-domain version of the HOPF bifurcation theorem [9].

The proposed proof generalizes to symmetric interconnections, as shown in the next section, and is expected to lead to a global version the result. In a forthcoming publication, we will also show that the OFP condition of Theorem 3.1 can be weakened using multipliers theory [12].

## IV. EXISTENCE AND STABILITY OF LIMIT CYCLES IN A NETWORK OF INTERCONNECTED OSCILLATORS

The analysis of Section III is now extended to a network of $N$ identical oscillators satisfying the assumptions of Theorem 3.1. Each oscillator is coupled to the rest of the network through an additionnal input which is chosen to be a linear combination of the oscillator outputs $y_j$ : $u_i = -y_{c_i} - \sum \gamma_{ij} y_j$. [3]

The interconnexion matrix $\Gamma$ is assumed to be symmetric, i.e. $\Gamma^T = \Gamma$. Let $k_0$ be a scalar such that $\Gamma' = \Gamma + k_0 I_{N\times N}$ is a symmetric positive semidefinite matrix of rank $q < N$ and define $k_p' = k_p + k_0$.

The network admits the representation illustrated in Figure 3 which is a MIMO extension of the block diagram in Figure 1. The MIMO linear blocks have the same passivity properties as in Section II as parallel connections of passive systems.

[3] The indice $i$ refers to the $i^{th}$ oscillator and the quantities $u$, $y$ and $y_c$ are defined for one oscillator in Figure 1. $\gamma_{ij}$ denotes the $ij^{th}$ element of the interconnexion matrix $\Gamma$.

The coupling matrix is included in the static nonlinearity which is therefore no longer diagonal. However, the static block can still be expressed as the sum of a strictly output passive map diag$\{\phi(\Sigma)\}Y + \Gamma'Y$ and a strictly active map $-k_p'Y$.

As a result the MIMO dissipativity characterization of the block diagram in Figure 3 is rigorously identical to the (SISO) dissipativity characterization of the block diagram in Figure 1 in spite of the coupling of the oscillators. This property is exploited in the next theorem to extend the result of Section III.

*Theorem 4.1:* Consider the feedback interconnexion $\Pi_{MIMO}$ of Figure 3 where the notation have been defined in Theorem 3.1. Denote by $M_{MIMO}$ the linearization of $\Pi_{MIMO}$ at the origin. Then,

**1)** There exists a critical value $k_p^* \geq 0$ of the parameter $k_p'$ such that all the eigenvalues of $M_{MIMO}$ have non positive real parts for $k_p' \leq k_p^*$ and that two nonzero complex conjugate eigenvalues of multiplicity $N - q$ each cross the imaginary axis at $k_p' = k_p^*$.

**2)** If no more than two eigenvalues of $M_{MIMO}$ (perhaps of high multiplicity) are on the imaginary axis at $k_p' = k_p^*$, then the system $\Pi_{MIMO}$ undergoes an equivariant HOPF bifurcation at $k_p' = k_p^*$. If $H$ is OFP($k_p^*$) this bifurcation is supercritical and gives rise to a locally stable limit cycle for $k_p' \gtrsim k_p^*$.

*Proof:*

The proof consists in two parts. First we generalize the root locus argument of Theorem 3.1. Then we prove that an equivariant HOPF bifurcation exists at $k_p' = k_p^*$ and that this bifurcation is supercritical.

**<u>Part 1)</u>**

The LAPLACE transform of the input-output equation for the MIMO linearized system is

$$Y(s) = Q(s) I_{N\times N} \left( Ext(s) + k_p' Y(s) - \Gamma' Y(s) \right) \quad (2)$$

where $Q(s)$ is the transfer function of the linearized system of one oscillator as defined in Figure 2.

The poles of the MIMO closed-loop transfer function are the complex values of $s$ such that

$$rank \left[ \frac{1 - k_p' Q(s)}{Q(s)} I_{N\times N} + \Gamma' \right] < N$$

Because $\Gamma' \geq 0$ is a symmetric matrix of rank $q$, there exists an orthogonal matrix $U$ such that $\Gamma' = U^T \Lambda U$ where $\Lambda = diag(\underbrace{0, \ldots, 0}_{N-q}, \lambda_{N-q+1}, \ldots, \lambda_N)$ with $0 < \lambda_{N-q+1} \leq \cdots \leq \lambda_N$.

We thus have to search for the complex values of $s$ rendering the diagonal matrix $\left[ \frac{1-k_p' Q(s)}{Q(s)} I_{N\times N} + \Lambda \right]$ singular. This matrix is singular for the complex values of $s$ solutions of one of the equations $\frac{1-(k_p' - \lambda_i) Q(s)}{Q(s)} = 0$, $i = 1, \ldots, N$. Thus its complex solutions are $\mu_j|_{k_p' := k_p' - \lambda_i}$[4] where $\mu_j$ are the poles of $R(s)$ (the transfer function associated with one isolated oscillator).

Considering these complex values of $s$ for $i = 1, \ldots, N$ we know from part 1) of Theorem 3.1 and $k_p' - \lambda_i \leq k_p'$, $i = 1, \ldots, N$ that

(1) they all have non positive real parts for $k_p' \leq k_p^*$;

(2) at least 2 eigenvalues each of multiplicity $N - q$ lie on the imaginary axis for $k_p' = k_p^*$ .

**Part 2)**

If only 2 conjugate eigenvalues of multiplicity $N - q$ lie on the imaginary axis for $k_p' = k_p^*$, $\Pi_{MIMO}$ undergoes an equivariant HOPF bifurcation [4]. The multiplicity of the two complex conjugate eigenvalues crossing the imaginary axis at $k_p' = k_p^*$ depends on the rank of $\Gamma'$.

A center manifold argument similar to the one used in Theorem 3.1 proves that the bifurcation is supercritical [12]. ∎

## V. AN ILLUSTRATIVE EXAMPLE

### A. *A SISO oscillator*

Consider Figure 1 defining the feedback nonlinear system $\Pi$ where $P(s) = \frac{1}{s+1}$ and $\phi(\sigma) = y^2$. Obviously, this transfer function has relative degree one and is OFP(1) since it corresponds to the negative feedback interconnection of a simple integrator with a proportional gain equal to 1.

It is easy to show that the transfer function of the linearized system $R(s) = \frac{s}{s^2 + (1-k_p)s + 1}$ loses stability at $k_p = k_p^* = 1$. The poles of $R(s)$ are $\frac{(k_p - 1) \pm \sqrt{(k_p-1)^2 - 4}}{2}$.

From Theorem 3.1, we conclude that the feedback system $\Pi$ undergoes a supercritical HOPF bifurcation at $k_p = 1$ giving rise to a locally stable limit cycle for $k_p \gtrsim k_p^*$.

[4] $\mu_j|_{k_p' := k_p' - \lambda_i}$ means that we replace each occurence of $k_p'$ by $k_p' - \lambda_i$, $i = 1, \ldots, N$ in the expression of the $j$ poles $\mu_j$ of $R(s)$.

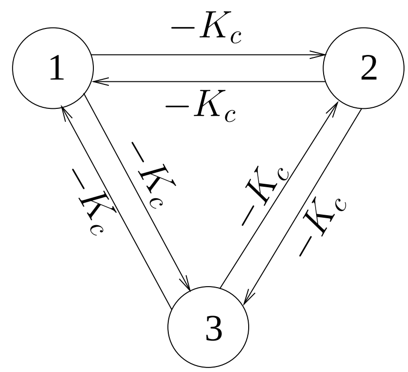


Fig. 4. Totally symmetric interconnection structure when 3 oscillators are interconnected.

### B. *Interconnection of oscillators*

Consider a network composed of $N$ identical oscillators constructed in Section V-A and connected to each other. Several kinds of interconnection structures are possible. As an illustrative example, we consider a very simple interconnection scheme : the totally symmetric interconnection structure.

In the totally symmetric interconnection structure, the oscillators in the network are all-to-all bilaterally coupled and the interconnection weights are identical. Let $K_c$ be the value of this unique interconnection weight. Thus for a totally symmetric interconnection structure we have

$$\Gamma = \begin{pmatrix} 0 & K_c & \cdots & K_c \\ K_c & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & K_c \\ K_c & \cdots & K_c & 0 \end{pmatrix}$$

with $K_c > 0$.

For $N = 3$, the totally symmetric interconnection structure is represented on Figure 4 where each circled number represents a SISO oscillator.

The eigenvalues of $\Gamma' = \Gamma + k_0 I_{N\times N}$ are $\lambda_1 = k_0 - K_c$ with multiplicity $(N-1)$ and $\lambda_2 = k_0 + (N-1)K_c$ with multiplicity 1. The value of $k_0$ such that $\Gamma'$ is positive semi-definite is $k_0 = K_c$. The rank of $\Gamma'$ is then obviously equal to 1.

From the eigenvalues of $\Gamma'$ we deduce the poles of the MIMO closed-loop transfer function. For each eigenvalue $\lambda_i$ of $\Gamma'$ we have to consider $\mu_j|_{k_p' := k_p' - \lambda_i}$, $i = 1, \ldots, N$ where $\mu_j$ are the poles of $R(s)$. The poles of the closed-loop transfer function are thus $\frac{(k_p'-1) \pm \sqrt{(k_p'-1)^2 - 4}}{2}$ with multiplicity $N - 1$ and $\frac{(k_p' - NK_c - 1) \pm \sqrt{(k_p' - NK_c - 1)^2 - 4}}{2}$ with multiplicity 1.

From Theorem 4.1 we conclude that the MIMO system representing the network undergoes a supercritical equivariant HOPF bifurcation at $k_p' = 1$. This is illustrated by the simulation results presented on Figure 5 where we have

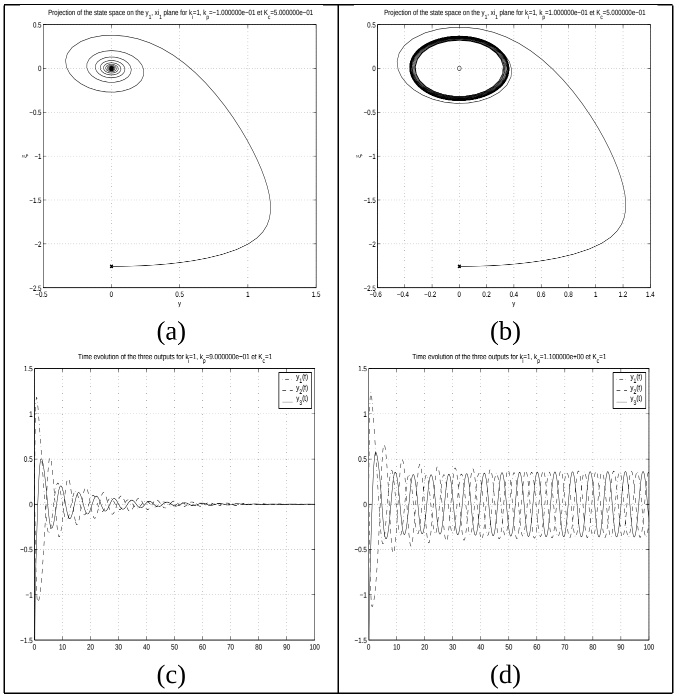


Fig. 5. State-space projected on the phase plane of one of the three oscillators (a) for $k'_p = 0.9$, (b) for $k'_p = 1.1$. Time evolution of the three outputs (c) for $k'_p = 0.9$, (d) for $k'_p = 1.1$.

plotted the projection of the state-space on the phase plane of one of the three oscillators and the time evolution of the outputs of the oscillators for $K_c = 1$.

## VI. CONCLUSION

This paper introduces generalizations of VAN DER POL and RAYLEIGH oscillators, which were regarded as feedback interconnections of two systems with particular input-output structure. This framework appears suitable to prove the existence of stable limit cycles both in higher-dimensional systems and in networks of coupled identical systems which retain the same input-output structure. As a first step, we characterized the existence of a supercritical HOPF bifurcation in these interconnexions.

We illustrated the existence and local stability of a limit cycle on a network of oscillators when a totally symmetric interconnexion structure is used. Simulations were carried out for a network of 3 such oscillators.

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