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Fast Consensus Via Predictive Pinning Control

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***Abstract*—By incorporating some predictive mechanism into a few pinning nodes, we show that convergence procedure to consensus can be substantially accelerated in networks of interconnected dynamic agents while physically maintaining the network topology. Such an acceleration stems from the compression mechanism of the eigenspectrum of the state matrix conferred by the predictive mechanism. This study provides a technical basis for the roles of some predictive mechanisms in widely-spread biological swarms, flocks, and consensus networks. From the engineering application point of view, inclusion of an efficient predictive mechanism allows for a significant increase in the convergence speed towards consensus.**

***Index Terms*—Consensus, multi-agent system (MAS), pinning control, predictive control, synchronization.**

I. Introduction

OVER the last decade, the collective motion of a group of autonomous agents (or particles) has been a subject of intensive research with potential applications in biology, physics, and engineering. One of the most remarkable characteristics of complex dynamical systems such as flocks of birds, schools of fish, or swarms of locusts, is the emergence of a state of collective order in which the agents reach a particular ordered state [1]–[3]. This distributed ordered state seeking problem can be further generalized to a consensus problem [4]–[7], where a group of self-propelled agents agree upon certain quantities of interest such as attitude, position, and so on. Solving consensus problems using distributed computational methods has direct implications on sensor network data fusion, load balancing, swarms/flocks, unmanned air vehicles (UAVs), attitude alignment of satellite clusters, congestion control of communication networks, multi-agent system (MAS) formation control, etc. [8]–[10].

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Convergence rate or speed is an important performance index in the analysis of consensus problems. Among the early works on consensus problems, Tsitsiklis [11] proposed a decentralized method to eliminate the disagreement within the group and hence derived the conditions for asymptotic convergence of each agent's decision sequence. In [4], Olfati-Saber and Murray found the relationship between the eigenvalue distribution of the Laplacian matrix L and its consensus performance. By this means, they have established the theoretical foundation of general consensus problems. To improve the speed of convergence towards consensus, they further proposed a method based on the addition of a few long links to a regular lattice, thus transforming it into a small-world network [12]. In [13], Xiao and Boyd transformed the fastest distributed linear averaging problem into a convex optimization problem by considering a particular per-step convergence optimization index, and designed an ultrafast consensus communication-weight assignment method for symmetric networks. In [14], Jadbabaie *et al.* derived consensus results for several direction alignment models including the well-known Vicsek model [1]. Specifically, the weak condition requiring linkage of the agents on some time intervals is proved to be sufficient for direction consensus. In Ren *et al.* [5], [6], the strongly connected condition guaranteeing consensus [4] is further relaxed into the existence of a rooted directed spanning tree over time. The most recent research includes the follower representative works: the existence of consensus behavior for a class of MASs was systematically addressed in [15], some necessary and sufficient conditions are provided in [16] for second-order consensus in multi-agent dynamic systems, a class of constrained consensus and optimization problems was studied in [17], a finite-time consensus protocol based on the Lyapunov method was given in [18], a linear quadratic regular (LQR)-based method is proposed in [19] and the outdated information is reused in [20], [21][34] to increase the convergence speed towards consensus.

In summary, most of the previous works focused on performance improvements, such as increasing the convergence speed towards consensus, improving the robustness to node and edge failures, or choosing proper interaction graphs possessing sufficiently strong algebraic connectivity to guarantee consensus, solely based on the information available at a given time instant in the network. It is noted that, in most of the former works, the prediction intelligence of each individual has been ignored. Each agent is only allowed to observe the current behavior of its neighbors and take a current movement decision based on this instantaneous observation. However, it is well accepted in the biology literature that living creatures typically possess some predictive intelligence allowing them to predict the future movements of their neighbors using their past observations. For example, as early as 1959, Woods [22] designed some bee swarm experiments and provided evidences for the

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existence of certain predictive mechanisms in bee swarm formation. In 1995, Montague *et al.* [23] proposed simple hebbian learning rules to explain the predictive mechanisms used by bees when foraging in uncertain environments. Apart from the investigation of the predictive mechanisms used during swarming and foraging, some researchers focused on the predictive functions of the optical and acoustical apparatuses of individuals inside bio-groups, especially the retina and the cortex [24]–[26]. Based on intensive experiments on the bio-eyesight systems, it was found that, when an individual observer's eyes prepare to follow a displacement of a visual stimulus, the visual form of adaptation was transferred from the current fixation to some future gaze position. These reported investigations support the conjecture of the existence of some predictive mechanisms inside many bio-groups. Promisingly, as an frontier engineering work, Farrai-Trecate *et al.* [27] have proposed a decentralized predictive mechanism to achieve consensus under some mild assumption, which provides a quite suitable starting point for applying predictive pinning control (PPC) to improve consensus performance. However, it focuses on addressing the input saturations and has not optimized the eigen-spectrum of the state matrix so as to accelerate the convergence procedure towards consensus, which is the essential technical point this paper addresses.

It is quite challenging to design an effective collective predictive mechanism to improve the synchronization/consensus performance of MASs. Let us explain the reason as follows: To gain an accurate prediction for each node, an agent must know the prediction strategy of its neighbors, and hence global information is required for the node. Thus, the controller becomes a centralized one, which contradicts the decentralized requirements of MASs. Fortunately, in recent years, more and more scholars have used pinning control to address the synchronization or consensus problems [28], [29]. In this methodology, just a small proportion of the nodes are selected as the pinning nodes as shown in Fig. 1, who know the global target of the entire MAS or can achieve the states of all the other nodes. By regulating these few pinning nodes, the synchronization or consensus procedure will be substantially accelerated at a low communication cost [28], [29]. In this paper, we propose some innovative solutions based on model predictive control (MPC), which is a widely used approach for its capability of handing the inter-individual coupling and the input/output constraints [30]–[32]. Fortunately, this pinning control framework can nicely act as a niche platform for predictive control of MASs, since the pinning nodes can be used to provide an accurate future state trajectory prediction due to the availability of the global MASs information at these nodes. In detail, we will show in this paper that, by incorporating one kind of predictive mechanism to these pinning nodes, the convergence speed towards consensus will be substantially increased. This observation allows us to reveal the roles of predictive mechanisms, which universally exist in natural bio-groups. It is also useful for engineering communication-limited MASs which converge much faster towards consensus.

The rest of this paper is organized as follows. In Section II, the two main problems addressed in this paper are formulated. In Section III, a predictive pinning control (PPC) protocol is presented together with its convergence analysis for networks with single-integrator dynamics. In Section IV, a PPC protocol for double-integrator networks is designed, which takes into account the states and their derivatives. Numerical simulation results showing PPC protocols' main characteristics and advantages are presented in Section V. Finally, conclusions are drawn in Section VI.

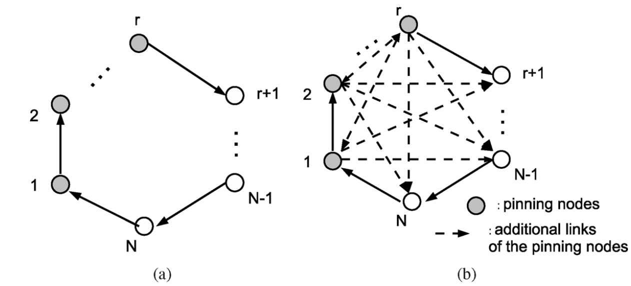

Fig. 1. (a) A network before pinning; (b) after pinning. Here, the pinning nodes $r \ll N$.

II. Problem Description

A digraph is denoted by $G = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{\nu_1, \ldots, \nu_N\}$ is the set of nodes, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. A weighted edge $a_{i,j}$ from node i to node j indicates the existence of a communication link, and A is the associated weighted *adjacency matrix*, i.e., $A = \{a_{i,j}\}_{i,j=1,\ldots,N} \in \mathbb{R}^{N\times N}_{\geq 0}$ with nonnegative elements $a_{i,j}$, which is zero when there is no communication link from i to j. Furthermore, we assume that there is no self-cycle, i.e., $a_{i,i} = 0$, $\forall i = 1, \ldots, N$.

Hereafter, we will focus on the digraphs with $r \ll N$ pinning nodes as shown in Fig. 1. (Note that, the other $N - r$ common nodes do not necessarily have the capability to obtain the state information from the pinning nodes.) Here, we define *pinning nodes* as the ones who are always connected to all the other nodes (including the other pinning nodes), i.e., $a_{i,j} > 0$ for $i \in \mathcal{V}_p$, $j \in \mathcal{V}$ and $j \neq i$. Here, $\mathcal{V}_p$ is set of the r pinning nodes. Therefore, pinning nodes in this paper are defined to be the ones who can always obtain the state information $x(t)$ from all the other nodes. In the remainder of the paper, without loss of generality, we set $\mathcal{V}_p = \{\nu_1, \ldots, \nu_r\}$. Of course, the r pinning nodes can also be deemed as "leaders" who always have connections to all the other nodes.

Let $x_i \in \mathbb{R}$ denote the state of node i. Generally, we say that the nodes of a network have reached *consensus* if and only if $x_i = x_j$ for all $\nu_i, \nu_j \in \mathcal{V} (i \neq j)$ [4]. We will address the consensus speeding up problems concerning networks with single- and double-integrator dynamics, where $r \ll N$.

1) *Single-integrator networks* The dynamics of a network of discrete-time integrator agents is defined by [4]:

$$x(k+1) = P_\epsilon x(k) \tag{1}$$

with $P_\epsilon = I_N - \epsilon L$, $L = \{l_{ij}\}_{i,j=1,\ldots,N} \in \mathbb{R}^{N\times N}$ denoting the *graph Laplacian matrix* induced by the topology G and being defined as $l_{ii} = \sum_{l\neq i}^{N} a_{il}$, $\forall i = 1, \ldots, N$ and $l_{ij} = -a_{ij}$, $\forall i \neq j$. Here, ϵ denotes the sampling period or step-size, and $I_N \in \mathbb{R}^{N\times N}$ is the identity matrix of dimension N.

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2) *Double-integrator networks* The discrete-time closed-loop dynamics is provided as [5]

$$z(k+1) = \bar{P}_\epsilon z(k) \tag{2}$$

with $z(k) := [x(k)^T, x(k+1)^T]^T$, $\bar{P}_\epsilon = \begin{bmatrix} \mathbf{0}_N & I_N \\ -I_N - \epsilon^2 L + \gamma\epsilon L & 2I_N - \gamma\epsilon L \end{bmatrix}$ (see [5]), and $\mathbf{0}_N = 0 \cdot I_N$.

The consensus protocols (1) and (2) ensure global asymptotic convergence to consensus as given in the following two lemmas respectively.

Lemma 1: [6] For system (1), the consensus situation

$$\lim_{k\to\infty} x(k) = \mathbf{1}\mu^T x(0) \tag{3}$$

where μ is the left eigvenvector of L corresponding to eigenvalue 0 who satisfies $\mu^T \mathbf{1}_N = 1$ and $\mathbf{1}_N := [1, \ldots, 1]^T_{N\times 1}$ (in particular, $\mu = \mathbf{1}_N/N$ for balanced networks, where $L^T\mathbf{1} = L\mathbf{1} = \mathbf{0}$ and $\mathbf{0} := 0 \cdot \mathbf{1}$), can be reached if and only if the sampling period $\epsilon \in (0, 1/d_{\max})$ and the following assumption is fulfilled.

A1: P_ϵ has one eigenvalue at 1 (or the network has a rooted directed spanning tree over time), and all the other eigenvalues inside the unit circle.

Lemma 2: [16] For system (2), the consensus

$$\lim_{k\to\infty} x(k) = \mathbf{1}\mu^T x(0) + \epsilon k \mathbf{1}\mu^T \delta x(0) \tag{4}$$

$$\lim_{k\to\infty} \delta x(k) = \mathbf{1}\mu^T \delta x(0) \tag{5}$$

where μ is the left eigvenvector of L corresponding to eigenvalue 0 who satisfies $\mu^T \mathbf{1}_N = 1$ and $\mathbf{1}_N := [1, \ldots, 1]^T_{N\times 1}$ (in particular, $\mu = \mathbf{1}_N/N$ for balanced networks, where $L^T\mathbf{1} = L\mathbf{1} = \mathbf{0}$), the velocity or state derivative vector $\delta x(k) := (x(k+1) - x(k))/(\epsilon)$ and $\lambda_k = \alpha_k + i\beta_k$ are the eigenvalues not equaling 1, can be reached if and only if the sampling period $\epsilon \in (0, 1/d_{\max})$, and

$$\gamma \geq \max_{2\leq k\leq N} \frac{\beta_k^2}{\alpha_k(\alpha_k^2 + \beta_k^2)} \tag{6}$$

and the following assumption is fulfilled.

A2: $\bar{P}_\epsilon$ (see (2)) has two eigenvalues at 1 (or the network has a rooted directed spanning tree over time), and all the other eigenvalues inside the unit circle.

In the remainder of the paper, for digraph with $r \ll N$ pinning nodes as shown in Fig. 1(b), we seek to design a suitable predictive mechanism represented by an additional state matrix P_{PPC} (resp. $\bar{P}_{\text{PPC}}$), which will be added to P_ϵ in (1) (resp. $\bar{P}_\epsilon$ in (2)), so as to increase the convergence speed towards consensus. Here, the subscript "PPC" represents "predictive pinning control". Of course, P_{PPC} and $\bar{P}_{\text{PPC}}$ only have influence on the dynamics of the pinning nodes of the network, which correspond to the first r rows of P_ϵ and the $(N+1) \sim (N+r)$ rows of $\bar{P}_\epsilon$, respectively. Apparently, because of the existence of the r pinning nodes, the network considered in the rest of the paper always has a rooted directed spanning tree over time [5], and hence the consensus is naturally guaranteed.

III. PPC of Single-Integrator Networks

In order to improve the consensus performances, we replace the classical control protocol $u_i(k) = -\sum_{j=1}^N a_{ij}\Delta x_{i,j}(k)$ [4] by the following PPC consensus protocol:

$$u_i(k) = -\sum_{j=1}^{N} a_{ij}\Delta x_{i,j}(k) + v_i(k), i \in \mathcal{V}_p \subset \mathcal{V} \tag{7}$$

$$u_i(k) = -\sum_{j=1}^{N} a_{ij}\Delta x_{i,j}(k), i \notin \mathcal{V}_p \tag{8}$$

where $v_i(k)$ is an additional term representing the PPC action, and $\Delta x_{i,j}(k) = x_i(k) - x_j(k)$. Recall that $\mathcal{V}_p = \{\nu_1, \ldots, \nu_r \,|\, r \ll N\}$ are the pinning nodes, and that we seek to accelerate the consensus procedure by incorporating a predictive mechanism into $\mathcal{V}_p$, which always know the states of all the other nodes.

With this PPC protocol, the network dynamics are given by

$$x(k+1) = P_\epsilon x(k) + \psi v(k) \tag{9}$$

with $v(k) = [v_1(k), \ldots, v_r(k)]^T$ representing the PPC decision values for the pinning nodes $\mathcal{V}_p$ and $\psi = [I_{r\times r}, \mathbf{0}_{r\times(N-r)}]^T$. The PPC element $v(k)$ will be calculated by solving a specific receding-horizon optimization problem as described below.

Using the consensus protocol (9), the future network state can be predicted based on the current state value $x(k)$ as shown in the matrix at the bottom of the page. Here, the integers H_p and H_u represent the prediction and control horizons, respectively.

In this way, the future evolution of the network can be predicted H_p steps ahead, as

$$X(k+1) = P_X x(k) + P_U U(k) \tag{10}$$

$$\begin{aligned}
x(k+2) &= P_\epsilon^2 x(k) + P_\epsilon \psi v(k) + \psi v(k+1) \\
&\vdots \\
x(k+H_u) &= P_\epsilon^{H_u} x(k) + \textstyle\sum_{j=0}^{H_u-1}(P_\epsilon^{H_u-j-1}\psi v(k+j)) \\
x(k+H_u+1) &= P_\epsilon^{H_u+1} x(k) + \textstyle\sum_{j=0}^{H_u-2}(P_\epsilon^{H_u-j}\psi v(k+j)) + (P_\epsilon + I)\psi v(k+H_u-1) \\
&\vdots \\
x(k+H_p) &= P_\epsilon^{H_p} x(k) + \textstyle\sum_{j=0}^{H_u-2}(P_\epsilon^{H_p-j-1}\psi v(k+j)) + \textstyle\sum_{j=0}^{H_p-H_u} P_\epsilon^j \psi v(k+H_u-1).
\end{aligned}$$

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with

$$X^T(k+1) = [x^T(k+1), \dots, x^T(k+H_p)] \in \mathbb{R}^{1\times H_p N}$$
$$U^T(k) = [v^T(k), \dots, v^T(k+H_u-1)] \in \mathbb{R}^{1\times H_u r}$$

$P_X^T = [P_\epsilon^T, \dots, (P_\epsilon^{H_p})^T] \in \mathbb{R}^{N\times H_p N}$, and the matrix $P_U \in \mathbb{R}^{H_p N\times H_u r}$ has the following structure shown in (11) at the bottom of the page, with $\Psi = \mathrm{diag}\{\psi, \dots, \psi\}_{HuN\times H_u r}$ and ψ given in (9).

Bearing in mind the goal of consensus, i.e., eliminating the disagreement of all the individuals of the network, we first calculate the state derivative of agents i and j in the network, $m \in \{1, \dots, H_p\}$ steps ahead, using the operator

$$\begin{aligned}\Delta x_{i,j}(k+m) &:= x_i(k+m) - x_j(k+m)\\ &= e_{i,j}x(k+m)\end{aligned} \tag{12}$$

with $e_{i,j} := e_i - e_j \in \mathbb{R}^{1\times N}$ and $e_j := [0, \dots, 0, 1_{j^{\text{th}}}, 0, \dots, 0]$ where only the jth element is non-zero. Based on (12), the network state derivative vector m steps ahead can be defined by

$$\begin{aligned}&\Delta x(k+m)\\ &\quad := [\Delta x_{1,2}(k+m), \dots\\ &\qquad \Delta x_{1,N}(k+m), \Delta x_{2,3}(k+m), \dots\\ &\qquad \Delta x_{2,N}(k+m), \dots\\ &\qquad \Delta x_{N-1,N}(k+m)]^T \in \mathbb{R}^{N(N-1)/2\times 1}.\end{aligned}$$

Consequently, the future evolution of the network's state derivative can be predicted H_p steps ahead as follows:

$$\begin{aligned}\Delta x(k+1) &= ex(k+1)\\ &\vdots\\ \Delta x(k+H_p) &= ex(k+H_p)\end{aligned} \tag{13}$$

with

$$\begin{aligned}e &:= \left[e_{1,2}^T, \dots, e_{1,N}^T, e_{2,3}^T, \dots, e_{2,N}^T, \dots, e_{N-1,N}^T\right]^T\\ &\in \mathbb{R}^{N(N-1)/2\times N}.\end{aligned} \tag{14}$$

It then follows from (13) that

$$\begin{aligned}&\Delta X(k+1)\\ &\quad := [\Delta x(k+1)^T, \dots, \Delta x(k+H_p)^T]^T\\ &\quad = EX(k+1) = E(P_X x(k) + P_U U(k))\\ &\quad = P_{XE}x(k) + P_{UE}U(k) \in \mathbb{R}^{H_p N(N-1)/2\times 1}\end{aligned} \tag{15}$$

with $E := \mathrm{diag}(e, \dots, e) \in \mathbb{R}^{H_p N(N-1)/2\times H_p N}$, $P_{XE} := EP_X$ and $P_{UE} := EP_U$.

To solve the consensus problem, we first set the receding-horizon optimization index as

$$J(k) = \|\Delta X(k+1)\|_Q^2 + \|U(k)\|_R^2 \tag{16}$$

where Q and R are compatible real, symmetric, positive definite weighting matrices, and $\| * \|_Q^2 = *^T Q *$. For simplicity, the weighting matrices can be set as

$$Q = qI(q > 0) \quad \text{and} \quad R = I. \tag{17}$$

In the optimization index (16), the first term penalizes the state derivative between each pair of states over the future H_p prediction steps, while the second term penalizes the PPC control energy over the future H_u control steps. In order to minimize (16), we compute the values of $U(k)$ that yield $\partial J(k)/\partial U(k) = 0$ to obtain the optimal PPC action by

$$\begin{aligned}\partial J(k)/\partial U(k) = 2P_{UE}^T QP_{XE}x(k)&\\ +2(P_{UE}^T QP_{UE} + R)U(k) &= 0,\end{aligned}$$

thus $U(k) = -(P_{UE}^T QP_{UE} + R)^{-1}P_{UE}^T QP_{XE}x(k)$, and the first N entries of $U(k)$ are extracted as the optimal PPC action

$$v(k) = \breve{P}_{\mathrm{PPC}}x(k) \tag{18}$$

where $\breve{P}_{\mathrm{PPC}} = -[I_r, \mathbf{0}_r, \dots, \mathbf{0}_r]_{r\times H_u r} \cdot (P_{UE}^T QP_{UE} + R)^{-1}P_{UE}^T QP_{XE}$. The associated closed-loop dynamics can then be written as

$$x(k+1) = (P_\epsilon + P_{\mathrm{PPC}})x(k) \tag{19}$$

with $P_{\mathrm{PPC}} = \psi\breve{P}_{\mathrm{PPC}}$. We hereby give the necessary and sufficient conditions guaranteeing asymptotic convergence to consensus for the above proposed consensus PPC.

Recall the two essential Lemmas 1 and 2, which have established the foundation of consensus for single- and double-integrator networks, respectively. To guarantee Assumptions A1 and A2, we will propose a lemma to "peel" (or remove) the eigenvalues at 1 from the state matrix spectrum as follows. Note that the eigenvector(s) corresponding of the eigenvalue(s) at 1

$$P_U = \begin{bmatrix} I_N & & & \\ P_\epsilon & I_N & & \\ \vdots & & \ddots & \\ P_\epsilon^{H_u-1} & P_\epsilon^{H_u-2} & \dots & I_N \\ P_\epsilon^{H_u} & \dots & P_\epsilon^2 & P_\epsilon + I_N \\ \vdots & & \vdots & \vdots \\ P_\epsilon^{H_p-1} & \dots & P_\epsilon^{H_p-H_u+1} & \begin{matrix} P_\epsilon^{H_p-H_u} \\ +P_\epsilon^{H_p-H_u-1} \\ +\dots+I_N \end{matrix} \end{bmatrix} \times \Psi \tag{11}$$

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merely determines the consensus value, while the remaining eigenvalues regulate the convergence speed towards consensus.

Lemma 3: If a matrix $D \in \mathbb{R}^{mN\times mN}$ is such that $D\mathbf{1}_{mN} = \mathbf{1}_{mN}$ and W has m eigenvalues at 1 with a geometric multiplicity of 1, then the matrix $D - B_m B_m^T/N$ has the same eigenvalues as W except for m eigenvalues which are located at 0 instead of 1. Here, $B_m := \mathrm{diag}(\mathbf{1}_N, \ldots, \mathbf{1}_N)_{mN\times m}$.

Proof: Consider $\Gamma = [\Pi, (1)/(\sqrt{N})B_m]$ with $\Pi = \mathrm{null}(B_m B_m^T)$, where $\mathrm{null}(B_m B_m^T) \in \mathbb{R}^{mN\times(mN-m)}$ denotes the matrix formed by the $mN-m$ orthonormal column vectors of dimension mN that span the null space of the matrix $B_m B_m^T$. Since by construction Γ is an orthonromal matrix, the similarity transformation $\Gamma^T(D - B_m B_m^T/mN)\Gamma$ preserves the eigenvalues of $D - B_m B_m^T/N$. We therefore obtain

$$\begin{aligned}
&\Gamma^T\left(D - B_m B_m^T/N\right)\Gamma \\
&= \begin{bmatrix} \Pi^T D\Pi & \Pi^T D\frac{B_m}{\sqrt{N}} \\ \frac{B^T}{\sqrt{N}}D\Pi & \frac{B_m^T}{\sqrt{N}}D\frac{B_m}{\sqrt{N}} \end{bmatrix} \\
&\quad - \begin{bmatrix} \Pi^T D\frac{1}{N}B_m B_m^T\Pi & \Pi^T\frac{1}{N}B_m B_m^T\frac{\mathbf{1}}{\sqrt{N}} \\ \frac{\mathbf{1}^T}{\sqrt{N}}\frac{1}{N}B_m B_m^T\Pi & \frac{\mathbf{1}^T}{\sqrt{N}}\frac{1}{N}B_m B_m^T\frac{\mathbf{1}}{\sqrt{N}} \end{bmatrix} \\
&= \begin{bmatrix} \Pi^T D\Pi & \mathbf{0} \\ \star & I_m \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_m \end{bmatrix} \\
&= \begin{bmatrix} \Pi^T D\Pi & \mathbf{0} \\ \star & \mathbf{0}_m \end{bmatrix}
\end{aligned}$$

which implies that $D - B_m B_m^T/N$ has the same eigenvalues as $\Pi^T D\Pi$ plus m eigenvalues at 0. Finally, using a similar argument, we can easily show that $\Pi^T D\Pi$ has the same eigenvalues as D minus m eigenvalues at 1. □

Thereby, for the single-integrator case (resp. the double-integrator case) as shown in Lemmas 1 (resp. Lemma 2), one can set $m = 1$ (resp. $m = 2$) in Lemma 3 to "peel" the eigenvalues at 1, so as to focus on the investigation of the distribution of the remaining eigenvalues, which dictate the convergence speed towards consensus.

For conciseness, we hereafter give a new definition on matrix spectrum.

Definition 1:

$$\rho_m(D) := \rho\left(D - B_m B_m^T/N\right) \tag{20}$$

where $\rho(\cdot)$ denote the matrix spectrum radius [33].

Then $\rho_m(D)$ is the $(m+1)$-th largest 2-norm [33] of the mN eigenvalues of D provided that D has m eigenvalues at 1 with geometric multiplicity of 1 and all the other eigenvalues in the unit circle.

Now, we will show that $\rho_m(D)$ determines the convergence speed towards consensus. Consider the Jordan normal form of the matrix D, there exists a matrix $S \in \mathbb{R}^{mN\times(mN-m)}$ such that $DS = S\bar{J}$. Here, $\bar{J} \in \mathbb{R}^{(mN-m)\times(mN-m)}$ is a block diagonal matrix with each block being Jordan block corresponding to the $m(N-1)$ eigenvalues inside the unit circle. Thus, one can find a suitable nonsingular matrix $\Phi = [\Psi,\ S]$ with Ψ having linear independent columns, such that

$$D = \Phi\begin{bmatrix} J & 0 \\ 0 & \bar{J} \end{bmatrix}\Phi^{-1}$$

and hence

$$\begin{aligned}
0 &= \lim_{k\to\infty} D^k - \Phi\begin{bmatrix} J^k & 0 \\ 0 & \bar{J}^k \end{bmatrix}\Phi^{-1} \\
&= \lim_{k\to\infty} D^k - \Phi\begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix}\Phi^{-1} \\
&= \lim_{k\to\infty} D^k - RJ^k R^\dagger
\end{aligned} \tag{21}$$

where $R^\dagger$ denotes the upper s rows of Φ^{-1}. From the above calculation (21), one has that $\rho_m(D)$ is the spectral radius of $\bar{J}$, which characterizes the convergence speed of $\bar{J}^k \to 0$ as $k \to \infty$ and that of $x(k)$ towards the consensus value, in the sense that the smaller the $\rho_m(D)$, the faster the convergence speed.

On the other hand, regarding the eigenvalues of the matrix P_{PPC} in (18), we will demonstrate the fact that the eigenvalue of P_{PPC} corresponding to the trivial eigenvector $\mathbf{1}$ is 0, which will be summarized in the following lemma.

Lemma 4: Consider an N-node network whose dynamics are described by (19), and with the associated weighting matrices given by (17), then one has $P_{\mathrm{PPC}}\mathbf{1} = \mathbf{0}$.

Proof: It is clear that $P_\epsilon\mathbf{1} = \mathbf{1}$, which directly leads to $P_{XE}\mathbf{1}_{N\times1} = EP_X\mathbf{1}_{N\times1} = E[\mathbf{1}^T, \ldots, \mathbf{1}^T]^T = [0, \ldots, 0]^T \in \mathbb{R}^{H_p N(N+1)/2\times1}$, and hence one has $P_{\mathrm{PPC}}\mathbf{1} = \mathbf{0}$. In other words, the eigenvalue of P_{PPC} associated with the trivial eigenvector $\mathbf{1}$ is 0. □

Based on Lemmas 3 and 4, we are ready to propose the main theorem for single-integrator networks in (19) as follows so as to demonstrate the merits of the proposed PPC consensus protocol.

Theorem 1 (Spectrum Strict Compression Theorem-Single): Consider an N-node network $G = (\mathcal{V}, \mathcal{E}, A)$ whose dynamics are described by (19), and with the associated weighting matrices given by (17). One then has

$$\rho(P_\epsilon + P_{\mathrm{PPC}} - \mathbf{1}\mathbf{1}^T/N) < \rho(P_\epsilon - \mathbf{1}\mathbf{1}^T/N). \tag{22}$$

Proof: See Appendix A. □

Now, it is safe to conclude from Theorem 1 that, except for the simple eigenvalue at 1, the spectrum of P_ϵ is effectively compressed by P_{PPC}, and hence the convergence speed towards consensus is effectively increased [4], [5].

IV. PPC of Double-Integrator Networks

We now consider a set of N agents with double integrator dynamics (2), and replace this classical control protocol by the following PPC consensus protocol:

$$\begin{aligned}
u_i(k) = -\sum_{j=1}^{N} a_{i,j}(\Delta x_{i,j}(k) \\
+ \gamma\Delta\dot{x}_{i,j}(k)) + v_i(k) \\
i \in \mathcal{V}_p \subset \mathcal{V};
\end{aligned} \tag{23}$$

$$\begin{aligned}
u_i(k) = -\sum_{j=1}^{N} a_{i,j}(\Delta x_{i,j}(k) \\
+ \gamma\Delta\dot{x}_{i,j}(k)), i \notin \mathcal{V}_p
\end{aligned} \tag{24}$$

where $v_i(k)$ is an additional term representing the PPC action, $\Delta x_{i,j}(k) = x_i(k) - x_j(k)$ and $\Delta\dot{x}_{i,j}(k) := \delta x_i(k) - \delta x_j(k)$.

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Recall that $\mathcal{V}_p = \{\nu_1, \ldots, \nu_r\}$ are the pinning nodes, and $\delta x_i(k) := (x_i(k+1) - x_i(k))/(\epsilon)$.

With this PPC protocol, the network dynamics are given by

$$z(k+1) = \bar{P}_\epsilon z(k) + w\psi v(k) \tag{25}$$
$$x(k+1) = w^T \bar{P}_\epsilon z(k) + \psi v(k) \tag{26}$$

with $z(k) = [x^T(k), x^T(k+1)]^T$, $\psi = [I_{r\times r}, \mathbf{0}_{(N-r)\times r}]^T$, $w := [\mathbf{0}_N, I_N]^T$ and $v(k) = [v_1(k), \ldots, v_r(k)]^T$ representing the PPC decision values of $\mathcal{V}_p$. The PPC element $v(k)$ will be calculated by solving the receding-horizon optimization problem as described below.

Using the consensus protocol (25), the future state evolution of the network can be predicted based on the current state value $z(k)$ as shown in the matrix at the bottom of the page.

In this way, the future evolution of the network can be predicted H_p steps ahead as

$$Z(k+1) = \bar{P}_Z z(k) + \bar{P}_U U(k) \tag{27}$$

with

$$\begin{aligned} Z^T(k+1) &= [z^T(k+1), \ldots, z^T(k+H_p)] \in \mathbb{R}^{1\times 2H_pN} \\ U^T(k) &= [v^T(k), \ldots, v^T(k+H_u-1)] \in \mathbb{R}^{1\times H_u r}, \end{aligned}$$

$\bar{P}_Z = [\bar{P}_\epsilon^T, \ldots, (\bar{P}_\epsilon^{H_p})^T]^T \in \mathbb{R}^{2N\times 2H_pN}$, and $\bar{P}_U$ given by (45) in Appendix B.

Bearing in mind the goal of consensus, i.e., eliminating the disagreement of all the individuals of the network, we calculate the network state derivative vector $m \in \{1, \ldots, H_p\}$ steps ahead by

$$\begin{aligned} &\Delta z(k+m) \\ &\quad := [\Delta x_{1,2}(k+m), \ldots, \Delta x_{1,N}(k+m) \\ &\qquad \Delta x_{2,3}(k+m), \ldots, \Delta x_{2,N}(k+m), \ldots \\ &\qquad \Delta x_{N-1,N}(k+m)]^T \\ &\quad \in \mathbb{R}^{N(N-1)/2\times 1}. \end{aligned}$$

Therefore, the future evolution of the network's state derivative can be predicted H_p steps ahead as follows:

$$\begin{aligned} \Delta z(k+1) &= \bar{e} z(k+1) \\ &\vdots \\ \Delta z(k+H_p) &= \bar{e} z(k+H_p) \end{aligned} \tag{28}$$

with $\bar{e} = [\bar{e}_{1,2}^T, \ldots, \bar{e}_{1,N}^T, \bar{e}_{2,3}^T, \ldots, \bar{e}_{2,N}^T, \ldots, \bar{e}_{N-1,N}^T]^T \in \mathbb{R}^{N(N-1)/2\times 2N}$, $\bar{e}_{i,j} := \bar{e}_i - \bar{e}_j \in \mathbb{R}^{1\times 2N}$ and $\bar{e}_j := [0, \ldots, 0, 1_{j^{\text{th}}}, 0, \ldots, 0]_{1\times 2N}$. It then follows from (28) that

$$\begin{aligned} &\Delta Z(k+1) \\ &\quad := [\Delta z(k+1)^T, \ldots, \Delta z(k+H_p)^T]^T \\ &\quad = \bar{E} Z(k+1) = \bar{E}(\bar{P}_Z z(k) + \bar{P}_U U(k)) \\ &\quad = \bar{P}_{ZE} z(k) + \bar{P}_{UE} U(k) \in \mathbb{R}^{H_pN(N-1)/2\times 1} \end{aligned} \tag{29}$$

with $\bar{E} := \operatorname{diag}(\underbrace{\bar{e}, \ldots, \bar{e}}_{H_p}) \in \mathbb{R}^{H_pN(N-1)/2\times 2H_pN}$, $\bar{P}_{ZE} := \bar{E}\bar{P}_Z$ and $\bar{P}_{UE} := \bar{E}\bar{P}_U$.

To solve the consensus problem, we first set the receding-horizon optimization index that defines the PPC consensus problem as follows:

$$J(k) = \|\Delta Z(k+1)\|_Q^2 + \|U(k)\|_R^2 \tag{30}$$

where Q and R are the same as (17). In order to minimize (30), we calculate $\partial J(k)/\partial U(k) = 0$ to obtain the optimal PPC action as below

$$U(k) = -(\bar{P}_{UE}^T Q \bar{P}_{UE} + R)^{-1} \bar{P}_{UE}^T Q \bar{P}_{ZE} z(k)$$

and the first N entries of $U(k)$ are extracted to yield the optimal PPC action

$$v(k) = \tilde{P}_{\text{PPC}} z(k) \tag{31}$$

where $\tilde{P}_{\text{PPC}} = -[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r} \cdot (\bar{P}_{UE}^T Q \bar{P}_{UE} + R)^{-1} \bar{P}_{UE}^T Q \bar{P}_{ZE}$. Then, the associated closed-loop dynamics can then be written as

$$z(k+1) = (\bar{P}_\epsilon + \bar{P}_{\text{PPC}}) z(k) \tag{32}$$

with $\bar{P}_{\text{PPC}} = w\psi\tilde{P}_{\text{PPC}}$. Next, we will give necessary and sufficient conditions guaranteeing asymptotic convergence to consensus for the above proposed PPC consensus protocol. Analogous to the single-integrator scenario, it is necessary to introduce two lemmas in advance.

Lemma 5: Consider an N-node network whose dynamics are described by (32), and with the compatible weighting matrices given by (17), then one has $\bar{P}_{\text{PPC}}\mathbf{1}_{2N} = \mathbf{0}_{2N}$.

Proof: First, we have by definition of $\bar{P}_\epsilon$ in (2) that

$$\bar{P}_\epsilon \mathbf{1}_{2N} = \mathbf{1}_{2N}. \tag{33}$$

$$\begin{aligned} z(k+2) &= \bar{P}_\epsilon^2 z(k) + \bar{P}_\epsilon w\psi v(k) + w\psi v(k+1) \\ &\vdots \\ z(k+H_u) &= \bar{P}_\epsilon^{H_u} z(k) + \textstyle\sum_{j=0}^{H_u-1} (\bar{P}_\epsilon^{H_u-j-1} w\psi v(k+j)) \\ z(k+H_u+1) &= \bar{P}_\epsilon^{H_u+1} z(k) + \textstyle\sum_{j=0}^{H_u-2} (\bar{P}_\epsilon^{H_u-j} w\psi v(k+j)) + (\bar{P}_\epsilon + I) w\psi v(k+H_u-1) \\ &\vdots \\ z(k+H_p) &= \bar{P}_\epsilon^{H_p} z(k) + \textstyle\sum_{j=0}^{H_u-2} (\bar{P}_\epsilon^{H_p-j-1} w\psi v(k+j)) + \textstyle\sum_{j=0}^{H_p-H_u} \bar{P}_\epsilon^j w\psi v(k+H_u-1). \end{aligned}$$

Combining (33) and the definition of $\bar{P}_Z$ in (27), we immediately obtain $\bar{P}_{ZE}\mathbf{1}_{2N\times 1} = E\bar{P}_Z\mathbf{1}_{2N\times 1} = E[\mathbf{1}^T,\ldots,\mathbf{1}^T]^T_{2H_pN\times 1} = [0,\ldots,0]^T \in \mathbb{R}^{H_pN(N-1)/2\times 1}$, which leads to $\bar{P}_{\text{PPC}}\mathbf{1} = \mathbf{0}$. In other words, the eigenvalue of $\bar{P}_{\text{PPC}}$ associated with the trivial eigenvector $\mathbf{1}$ is 0. □

Based on Lemma 5, we are ready to study the eigenvalues of the state matrix $\bar{P}_\epsilon + \bar{P}_{\text{PPC}}$ of the proposed PPC protocol (32) as below.

Lemma 6: Both the state matrix $\bar{P}_\epsilon$ in (2) and the PPC state matrix $\bar{P}_\epsilon + \bar{P}_{\text{PPC}}$ in (32) have two eigenvalues at 1 with a geometric multiplicity of 1, respectively.

Proof: First, we look at the eigenvalues at 1 of $\bar{P}_\epsilon$

$$\begin{aligned}&\det(\lambda I_{2N} - \bar{P}_\epsilon)\\ &\quad= \det((\lambda-1)^2 I_N + (\lambda\gamma\epsilon - \epsilon^2 - \gamma\epsilon)L)\\ &\quad= \Pi_{i=1}^N((\lambda-1)^2 - (\lambda\gamma\epsilon - \epsilon^2 - \gamma\epsilon)\varrho_i)\end{aligned}$$

where ϱ_i is the ith eigenvalue of $-L$. Therefore, it follows that $\lambda_{i+,-} = (2 + \gamma\epsilon\varrho_i \pm \sqrt{\epsilon^2\varrho_i(\gamma^2\varrho_i - 4)})/2$. Since $-L$ has one simple eigenvalue at 0, it follows that $\bar{P}_\epsilon$ has two eigenvalues at 1 with a geometric simplicity of 1.

Regarding the eigenvalues of $\bar{P}_\epsilon + \bar{P}_{\text{PPC}}$, it follows from (2) that $\bar{P}_\epsilon[\mathbf{1}_N^T, \mathbf{0}_N^T]^T = [\mathbf{0}_N^T, -\mathbf{1}_N^T]^T$, and hence it comes from the definition of $\bar{P}_{ZE}$ in (29) that $\bar{P}_{\text{PPC}}[\mathbf{1}_N^T, \mathbf{0}_N^T]^T = \mathbf{0}_{2N}^T$, which immediately leads to

$$\begin{aligned}&\begin{bmatrix}-\mathbf{1}_N & \mathbf{1}_N\\ -\mathbf{1}_N & \mathbf{0}_N\end{bmatrix}\cdot\begin{bmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{bmatrix}\\ &\quad= (\bar{P}_\epsilon + \bar{P}_{\text{PPC}})\begin{bmatrix}-\mathbf{1}_N & \mathbf{1}_N\\ -\mathbf{1}_N & \mathbf{0}_N\end{bmatrix}\\ &\quad= \begin{bmatrix}-\mathbf{1}_N & \mathbf{0}_N\\ -\mathbf{1}_N & -\mathbf{1}_N\end{bmatrix}\end{aligned}$$

with $\lambda_1 = 1$. Taking into consideration that $(\bar{P}_\epsilon + \bar{P}_{\text{PPC}})\mathbf{1}_{2N} = \mathbf{1}_{2N}$ in Lemma 5, one has that $\left[\begin{smallmatrix}-\mathbf{1}_N & \mathbf{1}_N\\ -\mathbf{1}_N & \mathbf{0}_N\end{smallmatrix}\right]$ is a Jordan eigenvector block associated with the eigenvalue $\lambda_1 = 1$ of $\bar{P}_\epsilon + \bar{P}_{\text{PPC}}$, and hence $\bar{P}_\epsilon + \bar{P}_{\text{PPC}}$ has two eigenvalues at 1 with a geometric simplicity of 1. □

Based on Lemmas 3, 5 and 6, we are now ready to provide a necessary and sufficient condition for consensus convergence of the proposed PPC protocol (32).

Theorem 2: For an N-node network whose dynamics are described by (32), and with the associated weighting matrices given by (17) and γ satisfying (6), the system state $x(k)$ and state derivative $\delta x(k)$ satisfy

$$\lim_{k\to\infty} x(k) = \mathbf{1}\mu^T x(0) + \epsilon k\mathbf{1}\mu^T\delta x(0) \tag{34}$$

$$\lim_{k\to\infty} \delta x(k) = \mathbf{1}\mu^T \delta x(0) \tag{35}$$

if and only if the following assumption holds

A3: $\rho(\bar{P}_\epsilon + \bar{P}_{\text{PPC}} - B_2\cdot B_2^T/N) < 1$,

noting that $\mu \in \mathbb{R}^N$ is the left eigvenvector of L corresponding to eigenvalue 0 who satisfies $\mu^T\mathbf{1}_N = 1$, and B_2 is given in Lemma 3. Particularly, for balanced N-node networks with $L^T\mathbf{1} = L\mathbf{1} = \mathbf{0}$, one has $\mu = (1/N)\mathbf{1}$.

Proof: See Appendix C. □

Therefore, it can be obtained from Theorem 2 that the state x and the state derivative δx asymptotically achieves the same consensus trajectories as the classical method shown in (4) and (5), once Assumption A3 and (6) are fulfilled. Now, we are at the point of characterizing the spectrum compression property of the proposed PPC algorithm (32), and thereby providing an explanation for its advantages over the classical algorithm (2).

Theorem 3 (Spectrum Strict Compression Theorem-Double): Consider an N-node network $G = (\mathcal{V}, \mathcal{E}, A)$ whose dynamics are described by (32), and with the compatible weighting matrices given by (17). One then has

$$\rho\left(\bar{P}_\epsilon + \bar{P}_{\text{PPC}} - B_2\cdot B_2^T/N\right) < \rho\left(\bar{P}_\epsilon - B_2\cdot B_2^T/N\right). \tag{36}$$

Proof: See Appendix D. □

It is concluded from the Theorem 3 and Lemma 3 that, except for the two eigenvalues at 1, the spectrum of $\bar{P}_\epsilon$ is effectively compressed by $\bar{P}_{\text{PPC}}$. Therefore, the convergence speed towards consensus is effectively increased [5], [6]. By this means, Theorem 3 provides an explanation as to why this PPC algorithm works in the case of double-integrator networks.

Remark 1: Since the global information of $\bar{P}_\epsilon$ and $x(k)$ is only available to the pinning nodes, Assumption A3 and (6) are hard to check in Theorem 2 in advance. Fortunately, they are not used in our controller design (31) but in verifying the parameter selections of γ, H_u, H_p and q after the implementation. The key point is (36) in Theorem 3, which implies that if the original control law (2) is convergent (in this case, (6) and Assumption A2 are fulfilled), then MPC protocol (32) is also convergent.

On the other hand, the technical analyses in Theorems 1, 2 and 3 are independent of the topology or the selection method of the pinning nodes. Admittedly, the optimal selection method of the pinning nodes according to the topology could be a challenging yet promising point deserving further investigation in our future work.

Remark 2: Note that the control law calculation complexity (see P_{MPC} of (18) and (31)) rises quickly along with increasing values of H_u and H_p, but it has been verified by numerical simulations that too small values of H_u and H_p will decrease the improvement in convergence speeds. Empirically speaking, selecting moderate values of H_u and H_p (like $H_u = 5$ and $H_p = 7$) is a good tradeoff between convergence speeds and computational complexities.

Remark 3: Although they have some common factors, the single-integrator case is not atrivial special case of double-integrator case. As shown in the proofs of Theorems 1–3, wehave used different techniques in the proofs to achieve the consensus values and to analyzethe spectrum compression mechanism, thus we have used two separated sections to propose them.

V. Case Studies

To illustrate the advantages of the PPC consensus protocol, we present simulation results comparing the convergence speeds obtained using the classical protocol given in (1) (resp. (2)) and the proposed PPC protocol given in (18) (resp. (31)) for single-integrator networks (resp. double-integrator networks) as below.

Without loss of generality, we consider a class of ring-shaped digraphs with r pinning nodes as shown in Fig. 1. The adjacency

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matrix A fulfills: 1) $a_{i,j} = 1$ with $i \in \mathcal{V}_p$, $j \in \mathcal{V}$ and $j \neq i$; 2) $a_{i,i+1} = 1$ with $i = 1, \ldots, N-1$; 3 $a_{N,1} = 1$; and 4) all the other entries of A are zeros. Since the objective is to reach consensus, the instantaneous disagreement index is typically set as $D(k) := \|x(k) - \mathbf{1}\mu^T \bar{x}(0)\|_2^2$.

A. *Case 1: Single-Integrator Networks*

Due to the similarity of using a linear quadratic regulator (LQR) to yield an optimal control law (see (16) and (30)), we also compare PPC with the LQR-based consensus algorithm (in abbreviation, LQR) proposed in [19] to demonstrate the superiority of PPC more vividly. As shown in the left panel of Fig. 2(a), the addition of the predictive mechanism defined in (18) yields a drastic increase in convergence speed $D(k)$ towards consensus. All the three methods (classical, LQR and PPC) can achieve the consensus value $\mathbf{1}\mu^T x(0) = \mathbf{1} \cdot 0.5363$ as shown in Fig. 2(b), where PPC achieves consensus the most quickly. The reason lies in the right panel of Fig. 2(a), where all the eigenvalues distributions of P_ϵ, P_{LQR} and $P_\epsilon + P_{\mathrm{PPC}}$ are exhibited. Obviously, the spectrum $\rho_1(P_\epsilon + P_{\mathrm{PPC}})$ is smaller than $\rho_1(P_{\mathrm{LQR}})$ and $\rho_1(P_\epsilon)$ (see Definition 1 for $\rho_m(\cdot)$). Moreover, consider the networks as shown in Fig. 1 with different pinning nodes numbers r, the spectrum $\rho_1(\cdot)$ of the corresponding state matrices of the classical, LQR and PPC laws are demonstrated in Table I. Hereby, Theorem 1 is verified, and the superiority of the PPC protocol is thus demonstrated.

It is noted that the advantage of PPC over LQR lies in the fact that PPC makes each node virtually link to the neighbor(s) of its neighbor(s) in future several steps, which increases the virtue connections of the network without adding physical links.

B. *Case 2: Double-Integrator Networks*

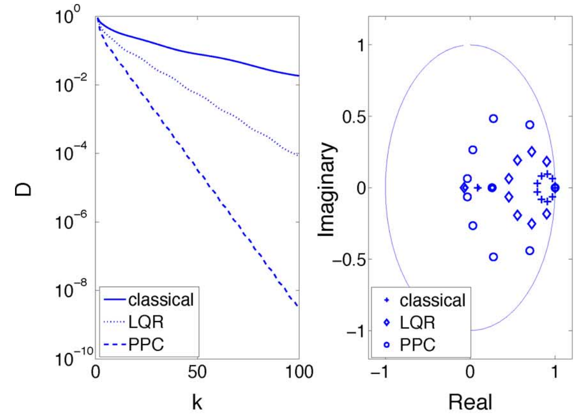
Bearing in mind the objective of reaching consensus as shown in (4) and (5), we set the instantaneous disagreement indexes of $x(k)$ and $\delta x(k)$ as

$$D_x(k) := \|x(k) - \mathbf{1}\mu^T x(0) - \epsilon k \mathbf{1}\mu^T \delta x(0)\|_2^2 \tag{37}$$
$$D_v(k) := \|\delta x(k) - \mathbf{1}\mu^T \delta x(0)\|_2^2 \tag{38}$$

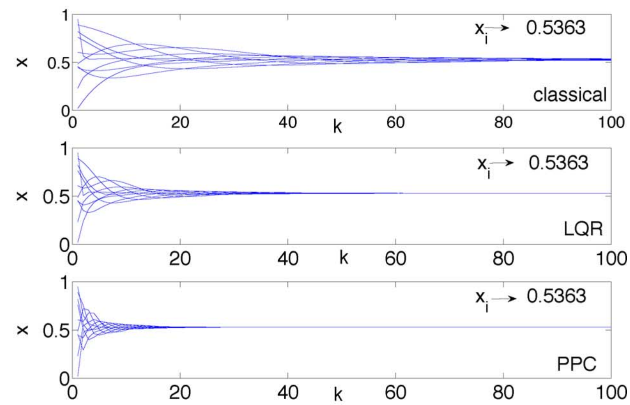
and then perform our simulation on the digraph shown in Fig. 1 with double-integrator dynamics.

As shown in the left and middle panels of Fig. 3(a), the addition of the predictive mechanism defined in (31) yields a drastic increase in convergence speed towards consensus of both state derivative $\delta x(k)$ (see Fig. 3(b)) and state $x(k)$ (see Fig. 3(c)). The improvement of the convergence speed roots in the eigenvalue distributions of $\bar{P}_\epsilon$ and $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ as shown in the right panel of Fig. 3(a). In particular, the third largest norm of the eigenvalues is $\rho_2(\bar{P}_\epsilon) = 0.9920$. Considering the controller (31) with $H_u = 3, H_p = 5$ and $q = 4$, one can find an optimal $\bar{P}_{\mathrm{PPC}}$ such that $\rho_2(\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}) = 0.8695$. By Theorem 2, we have $\lim_{k\to\infty} \delta x(k) = \mathbf{1} \cdot \mu^T \delta x(0) = 1.6590$. By this means, both Theorems 2 and 3 are exemplified, and the consensus acceleration is demonstrated through Fig. 3.

Remark 4: The time consumptions of 100 running steps in Fig. 2(b) are: classical-0.1532s and PPC-3.4583s The time consumptions of 200 running steps in Fig. 3(c) are: classical-0.5645s; PPC-12.1876s. Morever, as shown in Figs. 2(a) and 3(a), PPC has increase the convergence speed by more than 10 times in average. Therefore, taking into consideration the speed acceleration effect, one can deduce that the increased computational cost is reasonable and really worthy for systems with reasonable sizes. From an engineering point of view, even for large-scale systems, it still makes sense to speed up the consensus procedure at the cost of PPC's extra calculations, since agents' CPUs will become more and more powerful with the tremendous development of the IC technology. (Platform: Matlab 6.5, 2.8 G CPU and 2G RAM. The time consumption values are averages over 1000 independent runs.)



(a)



(b)

Fig. 2. (a) Left: consensus index $D_x(k)$; right: Eigenvalue distribution; (b) Network states trajectory achieving consensus. The parameters of PPC: $r = 1$, $N = 10$, $\epsilon = 0.1$, $H_u = 3$, $H_p = 5$, $q = 4$; the parameters of LQR [19]: optimal scaling factor $\beta = 0.83$, the weighting matrices $R = I$ and $Q = 3 \cdot I$; the initial states $x_i(0), i = 1, \ldots, N$ are selected randomly in $[0, 1]$.

VI. Conclusion

In this paper, we proposed a class of pinning predictive controllers (PPCs) for consensus networks to substantially increase their convergence speed towards consensus. The controller does not physically change the network topology or request additional communication channels. Its effectiveness and superiority have been demonstrated through theoretical analyses and numerical simulations.

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TABLE I
THE SECOND LARGEST NORM OF THE EIGENVALUES FOR DIFFERENT r

r	1	2	3	4	5
$\rho_1(P_\epsilon)$	0.9591	0.9527	0.9445	0.9335	0.9187
$\rho_1(P_{LQR})$	0.8793	0.8425	0.7732	0.7424	0.7192
$\rho_1(P_\epsilon + P_{PPC})$	0.7288	0.6947	0.6502	0.5922	0.5157

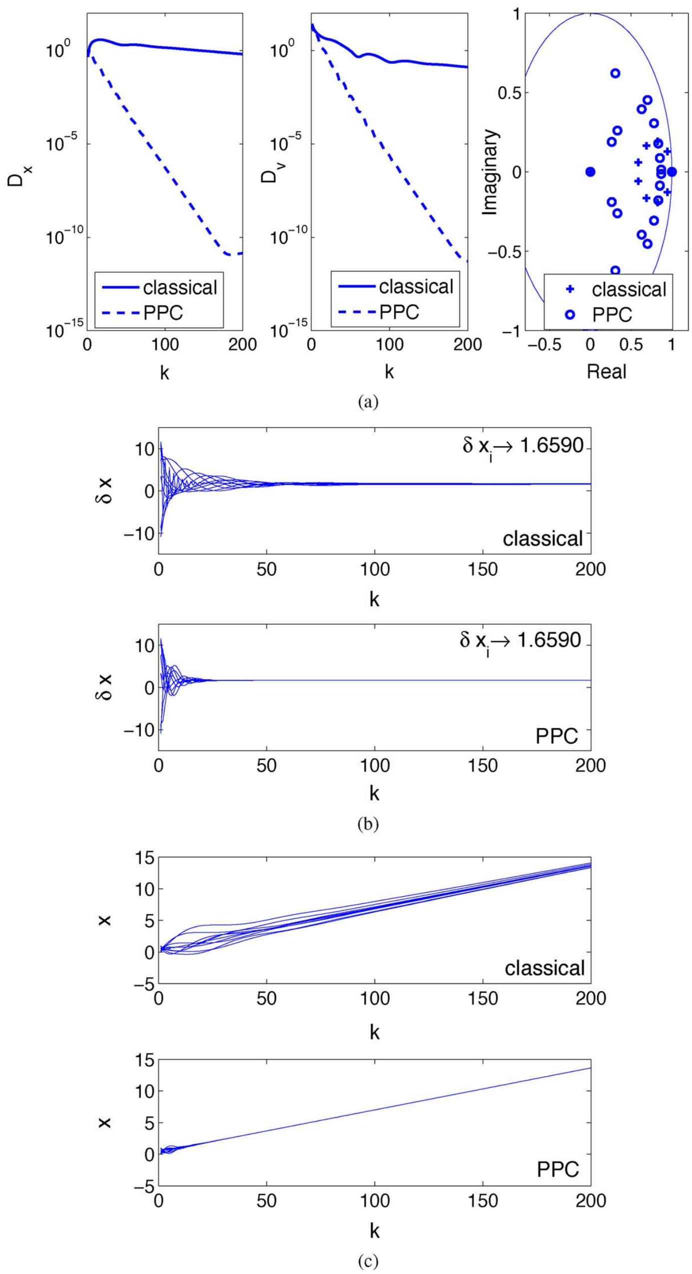

Fig. 3. (a) Left: state consensus index $D_x(k)$; middle: velocity consensus index $D_v(k)$; right: Eigenvalue distribution; (b) Network states trajectory achieving consensus; (c) Network velocities trajectory achieving consensus. The parameters: $r = 1, N = 10, \epsilon = 0.04, \gamma = 5, H_u = 3, H_p = 5, q = 4$, and the initial states $x_i(0), i = 1, \ldots, N$ are selected randomly in $[0, 1]$.

APPENDIX

A. Proof of Theorem 1

In order to prove this theorem, we first give a lemma on the spectra of symmetric positive semi-definite matrices as below.

Lemma 7: Consider an arbitrary symmetric, positive semi-definite matrix $D \in \mathbb{R}^{N\times N}$. Let λ be an eigenvalue of D with associated eigenvector η. One then has

1) $(D + I_N)^{-n}\eta = (1)/((\lambda+1)^n)\eta, \quad \forall n \in \mathbb{N}$;
2) $\|(D + I_N)^{-1}\|_2 \le 1$, where $\|E\|_2 := \sqrt{\lambda_{\max}(E^T E)}$

Proof: Property 1): Let λ and η be an arbitrary eigenvalue of D and its corresponding eigenvector, then $(1)/(\lambda+1)\eta = (D + I_N)^{-1}\eta$.

Since $D = D^T$, one has $((D + I_N)^{-1})^T(D + I_N)^{-1} = (D + I_N)^{-2}$, which directly leads to

$$\begin{aligned}
&((D + I_N)^{-1})^T(D + I_N)^{-1}\eta \\
&\quad = (D + I_N)^{-2}\eta \\
&\quad = \frac{1}{\lambda+1}(D + I_N)^{-1}\eta = \frac{1}{(\lambda+1)^2}\eta. \qquad (39)
\end{aligned}$$

Repeating the same argument for any $n \in \mathbb{N}$ proves property 1).

Property 2) is a direct consequence of (39). Using Property 1), one has $\|(D + I_N)^{-1}\|_2 = \sqrt{\lambda_{\max}((D + I_N)^{-2})}$. Furthermore, since D is positive semi-definite, one has $\lambda(D) \ge 0$ and $\|(D + I_N)^{-1}\|_2 = \sqrt{(1)/((\lambda_{\min}(D)+1)^2)} = (1)/(\lambda_{\min}(D) + 1) \le 1$, which completes the proof. □

Now we are ready to prove Theorem 1. Since $P_\epsilon \mathbf{1} = \mathbf{1}$ and $(P_\epsilon + P_{\mathrm{PPC}})\mathbf{1} = \mathbf{1}$, one has $P_{\mathrm{PPC}}\mathbf{1} = \mathbf{0}$. Hence

$$\begin{aligned}
P_{\mathrm{PPC}}P_\epsilon^{-1}\mathbf{1}\cdot\mathbf{1}^T &= P_{\mathrm{PPC}}P_\epsilon^{-1}P_\epsilon\mathbf{1}\cdot\mathbf{1}^T \\
&= P_{\mathrm{PPC}}\mathbf{1}\cdot\mathbf{1}^T = \mathbf{0}\cdot\mathbf{1}^T = \mathbf{0}
\end{aligned}$$

which immediately leads to

$$\begin{aligned}
&(I_N + P_{\mathrm{PPC}}P_\epsilon^{-1})(P_\epsilon - \mathbf{1}\mathbf{1}^T/N) \\
&\quad = P_\epsilon - \mathbf{1}\mathbf{1}^T/N + P_{\mathrm{PPC}} - P_{\mathrm{PPC}}P_\epsilon^{-1}\mathbf{1}\mathbf{1}^T/N \\
&\quad = P_\epsilon + P_{\mathrm{PPC}} - \mathbf{1}\mathbf{1}^T/N. \qquad (40)
\end{aligned}$$

Analogously, it is easily proven that

$$\mathbf{1}\cdot\mathbf{1}^T(P_\epsilon - \mathbf{1}\mathbf{1}^T/N) = 0. \qquad (41)$$

Substituting (41) into (40) yields

$$\begin{aligned}
&P_\epsilon + P_{\mathrm{PPC}} - \mathbf{1}\mathbf{1}^T/N \\
&\quad = (I_N + P_{\mathrm{PPC}}P_\epsilon^{-1})(P_\epsilon - \mathbf{1}\mathbf{1}^T/N) \\
&\quad = (I_N + P_{\mathrm{PPC}}P_\epsilon^{-1} - \mathbf{1}\mathbf{1}^T/N)(P_\epsilon - \mathbf{1}\mathbf{1}^T/N)
\end{aligned}$$

which implies

$$\begin{aligned}
&\rho(P_\epsilon + P_{\mathrm{PPC}} - \mathbf{1}\mathbf{1}^T/N) \\
&\quad \le \rho(I_N + P_{\mathrm{PPC}}P_\epsilon^{-1} - \mathbf{1}\mathbf{1}^T/N)\cdot\rho(P_\epsilon - \mathbf{1}\mathbf{1}^T/N). \qquad (42)
\end{aligned}$$

Thus, if $\rho(I_N + P_{\mathrm{PPC}}P_\epsilon^{-1} - \mathbf{1}\mathbf{1}^T/N) < 1$, one has that (36) holds. Accordingly, we start by proving that $\rho(I_N + P_{\mathrm{PPC}}P_\epsilon^{-1}) \le 1$. It is easy to see that

$$\begin{aligned}
&I_N + P_{\mathrm{PPC}}P_\epsilon^{-1} \\
&\quad = \psi[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}\left(P_{UE}^T Q P_{UE} + R\right)^{-1} \\
&\qquad \cdot \left(\left(P_{UE}^T Q P_{UE} + R\right)[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]^T\right. \\
&\qquad \left.\times\, \psi^T - P_{UE}^T Q P_{XE} P_\epsilon^{-1}\right).
\end{aligned}$$

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Taking into consideration that $P_{UE}[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]^T\psi = P_{XE}P_\epsilon^{-1}$, one thus has

$$\begin{aligned} I_N &+ P_{\mathrm{PPC}}P_\epsilon^{-1} \\ &= \psi[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}(P_{UE}^T Q P_{UE} + R)^{-1}R \\ &\cdot [I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]^T\psi^T \end{aligned} \tag{43}$$

which yields

$$\begin{aligned} \rho(I_N &+ P_{\mathrm{PPC}}P_\epsilon^{-1}) \\ &\le \|\psi\|_2 \cdot \|[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}\|_2 \\ &\cdot \left\|\left(P_{UE}^T Q P_{UE} + R\right)^{-1} R\right\|_2 \\ &\cdot \left\|[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}^T\right\|_2 \\ &\cdot \|\psi^T\|_2. \end{aligned}$$

Since $\|[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}\|_2 = 1$, $\|\psi\|_2 = 1$, and $P_{UE}^T Q P_{UE}$ is symmetric and positive semi-definite, Lemma 7 implies that

$$\rho\left(I_N + P_{\mathrm{PPC}}P_\epsilon^{-1}\right) \le 1. \tag{44}$$

On the other hand, it follows from $P_{\mathrm{PPC}}\mathbf{1} = \mathbf{0}$ and $P_\epsilon\mathbf{1} = \mathbf{1}$ that $(I_N + P_{\mathrm{PPC}}P_\epsilon^{-1})\mathbf{1} = \mathbf{1}$. Thus, $I_N + P_{\mathrm{PPC}}P_\epsilon^{-1}$ has a simple eigenvalue at 1, and hence it follows from Lemma 3 that $I_N + P_{\mathrm{PPC}}P_\epsilon^{-1} - \mathbf{1}\mathbf{1}^T/N$ has the same eigenvalues as $I_N + P_{\mathrm{PPC}}P_\epsilon^{-1}$ except for the eigenvalue at 1 which is now shifted to 0. Taking into consideration of (44), one has $\rho(I_N + P_{\mathrm{PPC}}P_\epsilon^{-1} - \mathbf{1}\mathbf{1}^T/N) < 1$, which immediately leads to (22) according to (42).

B. Predictive Matrix for Double-Integrator Networks

First, it is easy to see that the matrix $\bar{P}_U \in \mathbb{R}^{2H_pN\times H_u r}$ in (27) has the following structure

$$\bar{P}_U = \bar{P}_V W \tag{45}$$

with

$$\bar{P}_V = \begin{bmatrix} I_{2N} & & & \\ \bar{P}_\epsilon & I_{2N} & & \\ \vdots & & \ddots & \\ \bar{P}_\epsilon^{H_u-1} & \bar{P}_\epsilon^{H_u-2} & \cdots & I_{2N} \\ \bar{P}_\epsilon^{H_u} & \cdots & \bar{P}_\epsilon^2 & \bar{P}_\epsilon + I_{2N} \\ \vdots & & \vdots & \vdots \\ \bar{P}_\epsilon^{H_p-1} & \cdots & \bar{P}_\epsilon^{H_p-H_u+1} & \begin{matrix}\bar{P}_\epsilon^{H_p-H_u} \\ +\bar{P}_\epsilon^{H_p-H_u-1} \\ +\cdots + I_{2N},\end{matrix} \end{bmatrix}$$

and $W = \mathrm{diag}\{\underbrace{w\psi, \ldots, w\psi}_{H_u}\}$.

C. Proof of Theorem 2

{Sufficiency}: Since $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ has exactly two eigenvalues at 1 with geometric multiplicity of 1, it follows from Assumption A3 that the all the other eigenvalues of $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ except two ones at 1 are inside the unit circle. Letting $q = [q_a^T, q_b^T]^T$, where $q_a, q_b \in \mathbb{R}^N$, be an eigenvector of $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ associated with eigenvalue 1, and bearing in mind that $\bar{P}_{\mathrm{PPC}}q = q$, then we know that

$$\begin{aligned} (\bar{P}_\epsilon &+ \bar{P}_{\mathrm{PPC}})q \\ &= \bar{P}_\epsilon q \\ &= \begin{bmatrix} \mathbf{0}_N & I_N \\ -I_N - \epsilon^2 L + \gamma\epsilon L & 2I_N - \gamma\epsilon L \end{bmatrix}\begin{bmatrix} q_a \\ q_b \end{bmatrix} \\ &= \begin{bmatrix} q_a \\ q_b \end{bmatrix} \end{aligned}$$

which implies that $q_a = q_b$ and $Lq_a = 0$. That is, q is an eigenvalue of L associate with eigenvalue 0.

Note that $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ can be written in a Jordan canonical form as

$$\begin{aligned} \bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}} &= PJP^{-1} \\ &= [w_1, \ldots, w_{2N}]\begin{bmatrix} 1 & 1 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J' \end{bmatrix}\begin{bmatrix} v_1^T \\ \vdots \\ v_{2N}^T \end{bmatrix} \end{aligned}$$

where J' is the Jordan upper diagonal block matrix correspoding to non-one eigenvalues λ_{i+} and λ_{i-}, $i = 2, \ldots, N$.

Without loss of generality, we choose $w_1 = [\mathbf{1}_N^T, \mathbf{1}_N^T]^T$ and $w_2 = [-\mathbf{1}_N^T, \mathbf{0}_N^T]^T$, where it can be verified that w_1 and w_2 are respectively an eigenvector and generalized eigenvector of $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ associated with eigenvalue 1. Noting that $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ has exactly two eigenvalues at 1, which in turn implies that there exists a non-negative vector μ such that $\mu^T L = 0$ and $\mu^T\mathbf{1} = 1$ as shown in [6]. It can be verified that $v_1 = [\mu^T, \mathbf{0}^T]$ and $v_2 = [-\mu^T, \mu^T]$ are a generalized left eigenvector and left eigenvector of $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ associated with eigenvalue 1, respectively, where $v_1^T w_1 = 1$ and $v_2^T w_2 = 1$. Noting that all the eigenvalues except the two eigenvalues at 1 are inside the unit circle, we see that

$$\begin{aligned} (\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}})^k &= PJ^kP^{-1} \\ &= P\begin{bmatrix} 1 & k & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J'^k \end{bmatrix}P^{-1} \end{aligned}$$

which converges to $\begin{bmatrix} (1-k)\mathbf{1}\mu^T & k\mathbf{1}\mu^T \\ -k\mathbf{1}\mu^T & (k+1)\mathbf{1}\mu^T \end{bmatrix}$ for large running step k. Bearing in mind that, for large k

$$z(k) \to \begin{bmatrix} (1-k)\mathbf{1}\mu^T & k\mathbf{1}\mu^T \\ -k\mathbf{1}\mu^T & (k+1)\mathbf{1}\mu^T \end{bmatrix} z(0)$$

one has that (34) and (35) hold. Particularly, for balanced network, $\mu = 1/N\mathbf{1}$.

{Necessity}: Suppose that the sufficient condition of Assumption A3 does not hold, in other words, $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$ has either more than two eigenvalues at 1 or it has exactly two eigenvalues at 1 and at least one eigenvalue outside the unit circle. Without loss of generality, assume $\zeta_1 = \zeta_2 = 1$ and $\|\zeta_3\| > 1$, where $\zeta_m (m = 1, \ldots, 2N)$ denotes the mth eigenvalue of $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$. Letting $J = [j_{ml}]$ be the Jordan canonical form of $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}}$, we know that $j_{kk} = \zeta_m, (m = 1, \ldots, 2N)$. Then we see that $\lim_{k\to\infty} j_{mm}^k \ne 0, (m = 1, 2, 3)$, which in turn implies that the first three rows of $\lim_{k\to\infty}(\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}})^k$ are linearly independent. Therefore, one knows that the rank of $\lim_{k\to\infty} J^k$ is at least three, which implies that the rank of $\lim_{k\to\infty}(\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}})^k$

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is at least three. Note that the concensus is reached asymptotically if and only if $\lim_{k\to\infty}(\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}})^k = [\begin{smallmatrix}\mathbf{1}p^T\\ \mathbf{1}q^T\end{smallmatrix}]$, where $p, q \in \mathbb{R}^{N\times 1}$. As a consequence, the rank of $\lim_{k\to\infty}(\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}})^k$ cannot exceed two. This results in a contradiction. □

D. *Proof of Theorem 3*

Now we prove Theorem 3 as follows based on Lemma 7. Since $\bar{P}_\epsilon B_2 = [\mathbf{1}_{2N}, -\mathbf{1}_{2N}] + B_2$, one has $\bar{P}_\epsilon^{-1}B_2 = B_2 - \bar{P}_\epsilon^{-1}[\mathbf{1}_{2N}, -\mathbf{1}_{2N}] = B_2 - [\mathbf{1}_{2N}, -\mathbf{1}_{2N}]$, which leads to $\bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1}B_2 = \bar{P}_{\mathrm{PPC}}(B_2 - [\mathbf{1}_{2N}, -\mathbf{1}_{2N}]) = \mathbf{0}$. Therefore,

$$\begin{aligned}&(I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1})\left(\bar{P}_\epsilon - B_2B_2^T/N\right)\\ &= \bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}} - B_2B_2^T/N - \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1}B_2B_2^T/N\\ &= \bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}} - B_2B_2^T/N. \qquad (46)\end{aligned}$$

On the other hand, a suitable matrix Ω^* can be found such that

$$\left(\begin{bmatrix} I_N & \mathbf{0}_N\\ \mathbf{0}_N & \mathbf{0}_N\end{bmatrix} + \begin{bmatrix}\mathbf{0}_N & \mathbf{0}_N\\ \Omega^* & \mathbf{1}_N\mathbf{1}_N^T/N\end{bmatrix}\right)(\bar{P}_\epsilon - B_2B_2^T/N) = \mathbf{0}$$

making $\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}} - B_2B_2^T/N = (\bar{P}_\epsilon - B_2B_2^T/N)\Xi$ with $\Xi := I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1} - [\begin{smallmatrix} I_N & \mathbf{0}_N\\ \mathbf{0}_N & \mathbf{0}_N\end{smallmatrix}] - [\begin{smallmatrix}\mathbf{0}_N & \mathbf{0}_N\\ \Omega^* & \mathbf{1}_N\mathbf{1}_N^T/N\end{smallmatrix}]$. Therefore, if

$$\rho(\Xi) < 1 \qquad (47)$$

it follows immediately from (46) that

$$\begin{aligned}&\rho(\bar{P}_\epsilon + \bar{P}_{\mathrm{PPC}} - BB^T/N)\\ &= \rho(\Xi(\bar{P}_\epsilon - BB^T/N))\\ &\le \rho(\Xi)\rho(\bar{P}_\epsilon - BB^T/N) < \rho(\bar{P}_\epsilon - BB^T/N).\end{aligned}$$

Accordingly, we will start to prove (47). From $\bar{P}_{\mathrm{PPC}}$ in (32), one has that the first N rows of $\bar{P}_{\mathrm{PPC}}$ are zeros, which implies that the first N rows of $I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1} - [\begin{smallmatrix} I_N & \mathbf{0}_N\\ \mathbf{0}_N & \mathbf{0}_N\end{smallmatrix}]$ also equal $\mathbf{0}$. Besides, it is easy to see that

$$\begin{aligned}&I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1}\\ &= w\psi[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}(\bar{P}_{UE}^TQ\bar{P}_{UE} + R)^{-1}\\ &\cdot\left(\left(\bar{P}_{UE}^TQ\bar{P}_{UE} + R\right)[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{H_u r\times r}^T\psi^Tw^T\right.\\ &\left. - \bar{P}_{UE}^TQ\bar{P}_{ZE}\bar{P}_\epsilon^{-1}\right).\end{aligned}$$

Taking into consideration that $\bar{P}_{UE}[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]^T\psi^Tw^T = \bar{P}_{ZE}\bar{P}_\epsilon^{-1}$, one thus has

$$\begin{aligned}&I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1}\\ &= w\psi[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}(\bar{P}_{UE}^TQ\bar{P}_{UE} + R)^{-1}R\\ &\cdot[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]^T\psi^Tw^T \qquad (48)\end{aligned}$$

which yields

$$\begin{aligned}&\rho\left(I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1}\right)\\ &\le \|w\|_2\|\psi\|_2\|[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}\|_2\\ &\cdot\left\|\left(\bar{P}_{UE}^TQ\bar{P}_{UE} + R\right)^{-1}R\right\|_2\\ &\cdot\|[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]^T\|_2\|\psi\|_2\|w\|_2.\end{aligned}$$

Since $\|[I_r, \mathbf{0}_r, \ldots, \mathbf{0}_r]_{r\times H_u r}\|_2 = \|w\|_2 = \|\psi\| = 1$, and note that $\bar{P}_{UE}^TQ\bar{P}_{UE}$ is symmetric and positive semi-definite, it follows from Lemma 7 that

$$\rho(I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1}) \le 1. \qquad (49)$$

Meanwhile, it is obtained from $\bar{P}_{\mathrm{PPC}}\mathbf{1}_{2N} = \mathbf{0}_{2N}$ and $\bar{P}_\epsilon\mathbf{1}_{2N} = \mathbf{1}_{2N}$ that $(I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1})\mathbf{1}_{2N} = \mathbf{1}_{2N}$. Analogously, one has $\mathbf{1}_{2N}^T(I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1}) = \mathbf{1}_{2N}^T$. Thus, the right-bottom $N\times N$ block of

$$I_{2N} + \bar{P}_{\mathrm{PPC}}\bar{P}_\epsilon^{-1} - \begin{bmatrix} I_N & \mathbf{0}_N\\ \mathbf{0}_N & \mathbf{0}_N\end{bmatrix}$$

has just a simple eigenvalue at 1 and all the other eigenvalues inside the unit circle. Bearing in mind Theorem 1, one has that (47) holds. This completes the proof. □

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