Global State Synchronization in Networks of Cyclic Feedback Systems

Abdullah Hamadeh, Guy-Bart Stan, Rodolphe Sepulchre, and Jorge Gonçalves

***Abstract*—This technical note studies global asymptotic state synchronization in networks of identical systems. Conditions on the coupling strength required for the synchronization of nodes having a cyclic feedback structure are deduced using incremental dissipativity theory. The method takes advantage of the incremental passivity properties of the constituent subsystems of the network nodes to reformulate the synchronization problem as one of achieving incremental passivity by coupling. The method can be used in the framework of contraction theory to constructively build a contracting metric for the incremental system. The result is illustrated for a network of biochemical oscillators.**

***Index Terms*—Incremental dissipativity, networks of cyclic feedback biochemical oscillators, synchrony analysis.**

I. Introduction

Synchronization of dynamical systems is a commonly occurring phenomenon. It features in many biological networks, including that of neurons in the suprachiasmatic nucleus (SCN) of the hypothalamus, responsible for the generation of circadian rhythms in mammals. Mathematically, synchronization is a contraction property for the difference between the solutions of interconnected systems. Viewed in another way, we can determine whether two coupled systems synchronize by studying the asymptotic attractivity and stability of a synchronization manifold.

Several works have examined the *local* stability of the synchronization manifold. Transverse Lyapunov exponents [1] and master stability functions [2], [3] have been used to show that under certain coupling conditions the components of the trajectories transverse to the synchronization manifold are stable in a neighborhood of the manifold. Attractivity of the synchronization manifold can be studied using tools such as incremental stability [4] and contraction theory [5]–[7].

The key observation of [5]–[7] is that global asymptotic synchronization follows if the differences between corresponding states of coupled nodes (the *incremental states*) globally satisfy a contraction property. Using an algorithmic approach based on contraction theory, [8] determines whether general interconnected nodes will asymptotically synchronize. This is done by using the Gershgorin disk theorem to verify whether there exists a negative definite matrix measure of the Jacobian of the auxiliary system obtained using the approach of [7].

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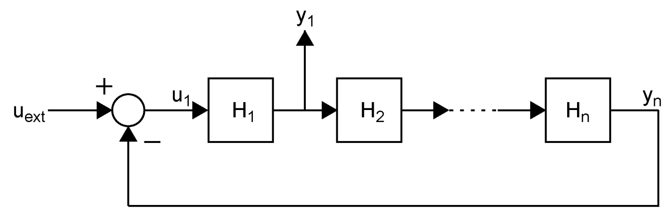


Fig. 1. Isolated node cyclic feedback system.

Contraction also follows from the construction of a Lyapunov function that operates on the incremental signals. Constructing such a Lyapunov function remains a challenging question in general. When the system to analyze consists of several interconnected subsystems, the construction of a Lyapunov function for the whole system from the storage functions of the (open) subsystems may be done using dissipativity theory [9], [10]. *Incremental dissipativity* provides a corresponding approach for constructing a Lyapunov function for contraction or incremental stability analysis [6], [11]–[13]. The design contribution of this technical note is to show that incremental dissipativity facilitates the construction of an incremental storage function that can be used to prove global asymptotic incremental stability (and hence global asymptotic state synchronization), as dissipativity does for global asymptotic stability. The constructive nature of the proposed approach makes the analysis easily scalable to networks of nodes of high dimension.

This synchronization methodology is illustrated for the case of cyclic feedback systems (CFS). CFS are typically used to model the dynamics of a chain of biochemical reactions where the final product inhibits the production of the first product in the chain whilst each intermediate product activates the subsequent reaction (see Fig. 1). The biological importance of CFS is discussed in [14], whilst [15] shows through simulations how networks of synchronizing CFS can model circadian timing in mammals. In [14], [16], it is shown that the *secant gain condition* provides a less conservative local stability certificate for CFS than the small gain theorem. In the more recent papers [17] and [18], this stability analysis is generalized to CFS composed of output strictly passive (OSP) subsystems. In particular, [18] shows that the secant gain condition is a necessary and sufficient condition for the diagonal stability of CFS. To find conditions for the global asymptotic synchronization of coupled CFS, we combine the results in [18] and [12] to exploit the CFS and construct an incremental storage function for the CFS network.

This technical note is structured as follows: Section II introduces some preliminaries concerning synchronization and incremental dissipativity. Section III characterizes CFS and gives sufficient conditions for their incremental dissipativity. In Section IV, we derive strong coupling conditions for synchronization in networks of CFS based on their incremental dissipativity properties and show the link between this technique, the contraction theory approach and master stability functions. The result is illustrated for a network of biochemical oscillators in Section V. We conclude with a discussion of the results.

II. Synchronization and Incremental Dissipativity

Consider a SISO system Υ represented by a state-space model of the form

$$\Upsilon \begin{cases} \dot{x} = \varphi(x,u), & x \in \mathbb{R}^r, u \in \mathbb{R} \\ y = \varrho(x), & y \in \mathbb{R} \end{cases} \tag{1}$$

where $u(t)$, $y(t)$, and $x(t)$ denote its input, output and state respectively. Let $x_a(t)$ and $x_b(t)$ be two solutions of Υ, with the corresponding input-output pairs $(u_a(t), y_a(t))$, and $(u_b(t), y_b(t))$. Denote by $\Delta x = x_a - x_b$, $\Delta u = u_a - u_b$, and $\Delta y = y_a - y_b$ the corresponding incremental variables. System (1) is incrementally dissipative if there exists a radially unbounded incremental storage function $S_\Delta(\Delta x) > 0 \forall \Delta x \neq 0$, with $S_\Delta(0) = 0$ and an incremental supply rate $w(\Delta u, \Delta y)$ such that, if $S_\Delta(\Delta x)$ is differentiable (i.e. $S_\Delta \in \mathcal{C}^1$)

$$\dot{S}_\Delta(\Delta x) \leq w\left(\Delta u(t), \Delta y(t)\right) \tag{2}$$

is satisfied for all time t and along any pair of trajectories $(x_a(t), x_b(t))$ (see [9] for a definition of dissipativity). Incremental dissipativity with incremental supply rate $w(\Delta u, \Delta y) = \Delta y \Delta u$ is called *incremental passivity*. Incremental dissipativity with the incremental supply rate $w(\Delta u, \Delta y) = -(\Delta y)^2 + \gamma \Delta y \Delta u$ with $\gamma \in (-\infty, \infty)$ is called *incremental output feedback passivity* (iOFP($1/\gamma$)). When $\gamma > 0$ the system possesses an excess of incremental passivity and is said to be *incrementally output strictly passive* (iOSP). When $\gamma < 0$, the system has a shortage of incremental passivity and $-1/\gamma$ quantifies the minimum amount of proportional negative incremental output feedback required to make the system incrementally passive. Following the concept of 'secant gain' in [18], $\gamma > 0$ is called the *incremental secant gain*.

Remark 1: For linear systems, (output strict) passivity implies incremental (output strict) passivity and that the incremental secant gain equals the secant gain [11]. Passivity also implies incremental passivity for a monotone increasing, static nonlinearity: if $\phi(\cdot)$ is monotone increasing, then $(s_1 - s_2)(\phi(s_1) - \phi(s_2)) = \Delta s \Delta \phi(s) \geq 0$, $\forall \Delta s = s_1 - s_2$.

A system is incrementally dissipative if, given any two sets of initial conditions, inputs and corresponding outputs, (2) is satisfied. By extension, incremental dissipativity, a property of each node, can be used to analyze an entire network of interconnected copies of such a node. The main result linking iOFP of nodes of a network to output synchronization states that if each node is iOFP and the nodes are strongly coupled, then all the nodes will asymptotically synchronize [12], [19]. We shall show that a CFS is iOFP under mild assumptions (Theorem 1), and then use this result to prove asymptotic state synchronization in CFS networks (Theorem 2).

III. Incremental Output Feedback Passivity of CFS

The class of cyclic feedback systems (Fig. 1) typically arises in a sequence of biochemical reactions wherein the end product inhibits the rate of the first reaction while intermediate products activate subsequent reactions (see [14], [16], [18], [20], [21]). We assume each subsystem H_i in Fig. 1 is an iOSP system, with state $x_i \in \mathbb{R}^{r_i}$, input $u_i \in \mathbb{R}$ and output $y_i \in \mathbb{R}$, and has the state-space description

$$\dot{x}_i = \varphi_i(x_i, u_i), \qquad x_i \in \mathbb{R}^{r_i}, \quad u_i \in \mathbb{R} \tag{3}$$

$$y_i = \varrho_i(x_i), \qquad y_i \in \mathbb{R} \tag{4}$$

with $\varphi_i : \mathbb{R}^{r_i} \times \mathbb{R} \to \mathbb{R}^{r_i}$ and $\varrho_i : \mathbb{R}^{r_i} \to \mathbb{R}$, both being Lipschitz continuous functions. The structure of Fig. 1 imposes the input-output conditions

$$\begin{aligned} u_1 &= u_{\text{ext}} - y_n \\ u_i &= y_{i-1} \qquad i = 2, \cdots, n. \end{aligned} \tag{5}$$

The position of the external input u_{ext} with respect to the negative feedback $-y_n$ is arbitrary and is taken to be an input to H_1 without loss of generality. With u_{ext} as an input to H_1, the external input/output pair of this CFS are u_{ext} and y_1 respectively.[1] This choice of input-output pair will play a fundamental role in asymptotic synchronization, allowing us to prove iOFP of the CFS (see Theorem 1).

Remark 2: Note that the inputs and outputs of multiple subsystems H_i can be simultaneously used for coupling [22], though for simplicity we shall present the results for the case where only one coupling pair is used.

A. Incremental Secant Gain of Subsystems H_i

For iOSP systems of the form (3), (4), the incremental passivity of each block H_i can be quantified by its associated incremental secant gain. A specific form of the state-space descriptions (3), (4) of subsystems H_i typically used to represent chemical reactions (and in particular, reactions involving Michaelis-Menten and Hill terms) is given by

$$\dot{x}_i = -f_i(x_i) + u_i, \qquad u_i \in \mathbb{R}\ (\text{input}) \tag{6}$$

$$y_i = g_i(x_i), \qquad y_i \in \mathbb{R}\ (\text{output}) \tag{7}$$

where $f_i(\cdot) : \mathbb{R} \to \mathbb{R}$ and $g_i(\cdot) : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$ represent functions which are strictly increasing and Lipschitz continuous on $\mathbb{R}$. Consider two particular solutions $x_{i_a}(t), x_{i_b}(t)$ of (6), (7) and their associated input-output pairs u_{i_a}, y_{i_a} and u_{i_b}, y_{i_b}. With the conditions on $f_i(\cdot)$ and $g_i(\cdot)$ above, it follows that $0 < f_i'(x_i) < \infty$ and $0 < g_i'(x_i) < \infty$. If $\sup_{x_i}(g_i'(x_i)/f_i'(x_i))$ exists, and if $\gamma_i \triangleq \sup_{x_i}(g_i'(x_i)/f_i'(x_i)) > 0$ and

$$v_{i_{a,b}} \triangleq \gamma_i \int_{\mu_{i_a} - \mu_{i_b} = 0}^{\mu_{i_a} - \mu_{i_b} = x_{i_a} - x_{i_b}} \left(g_i\left(\mu_{i_a}\right) - g_i\left(\mu_{i_b}\right)\right) \mathrm{d}\left(\mu_{i_a} - \mu_{i_b}\right) \tag{8}$$

then if the integral (8) is well defined, $\dot{v}_{i_{a,b}}$ satisfies

$$\begin{aligned} \dot{v}_{i_{a,b}} &= -\gamma_i\left(g_i\left(x_{i_a}\right) - g_i\left(x_{i_b}\right)\right)\left(f_i\left(x_{i_a}\right) - f_i\left(x_{i_b}\right)\right) \\ &\quad + \gamma_i\left(g_i\left(x_{i_a}\right) - g_i\left(x_{i_b}\right)\right)\left(u_{i_a} - u_{i_b}\right) \\ &\leq -\left(y_{i_a} - y_{i_b}\right)^2 + \gamma_i\left(y_{i_a} - y_{i_b}\right)\left(u_{i_a} - u_{i_b}\right). \end{aligned}$$

From this inequality, we see that the block H_i is iOSP $(1/\gamma_i)$, with $\gamma_i = \sup_{x_i}(g_i'(x_i)/f_i'(x_i))$ being the incremental secant gain. If $\sup_{x_i}(g_i'(x_i)/f_i'(x_i))$ does not exist for $x_i \in \mathbb{R}$, it may nevertheless exist in a compact subset $\Omega_i \subset \mathbb{R}$ that is globally attractive and in which x_i is invariant, in which case $\gamma_i \triangleq \sup_{x_i \in \Omega_i}(g_i'(x_i)/f_i'(x_i))$. If the integral (8) is not well defined, the combined subsystem (6), (7) can be split into a cascade of the dynamic subsystem (6), with output x_i and the static subsystem (7) with input x_i. The secant gains for each subsystem are then given by $1/\inf_{x_i} f_i'(x_i)$ and $\sup_{x_i} g_i'(x_i)$ respectively.

B. Overview of the Results

From [18], if all the blocks H_i are output strictly passive (OSP) with a secant gain $\bar{\gamma}_i$, and, if a particular secant gain condition ($\bar{\gamma}_1 \cdots \bar{\gamma}_n < (\sec(\pi/n))^n$) is satisfied, then the CFS is OSP with respect to input u_{ext} and output y_1. As a corollary, if blocks H_i are iOSP with an incremental secant gain γ_i, and if the incremental secant gain condition $(\gamma_1 \cdots \gamma_n < (\sec(\pi/n))^n)$ is satisfied, then the CFS is iOSP with respect to u_{ext}, y_1. Even if the incremental secant gain condition is not satisfied, the CFS can be shown to be iOFP: by adding a proportional feedback of gain k around H_1 (Fig. 2), the feedback changes the effective incremental secant gain of H_1 to $\tilde{\gamma}_1 = \gamma_1/(1 +$

[1] As discussed in [22], if u_{ext} were chosen to appear as the external input of H_i, so that $u_1 = -y_n$, $u_i = u_{\text{ext}} + y_{i-1}$, $u_k = y_{k-1}$, $\forall k \neq 1, i$ then the external input/output pair of the corresponding CFS would be chosen to be u_{ext} and y_i respectively.

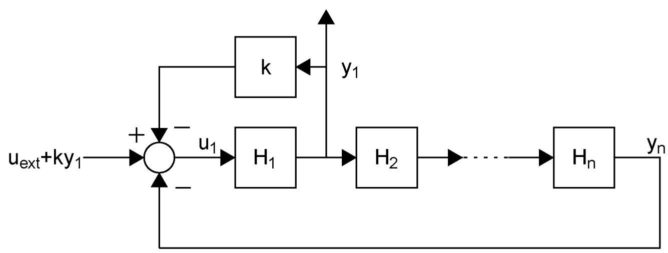


Fig. 2. CFS with negative feedback around H_1.

$k\gamma_1)$. By sufficiently increasing k, the incremental secant gain condition $\tilde{\gamma}_1\gamma_2\cdots\gamma_n < (\sec(\pi/n))^n$ can always be satisfied. Such a value of k quantifies the shortage of incremental passivity of the CFS node (Theorem 1). Strongly oupling several CFS to form a network effectively introduces an incremental negative feedback between the inputs and the outputs of the network components, compensating for the shortage of incremental passivity. This observation is used in Theorem 2 to find conditions for global asymptotic state synchronization.

C. Notation

We consider networks of N CFS of the form (3), (4), (5). As convention, $i = 1,\cdots,n$ will denote the index of block H_i in a given CFS while $j = 1,\cdots,N$ will denote the index of a particular CFS in the network.

Defining $R = \sum_{i=1}^n r_i$, the vector of the states (respectively, outputs) of the j^{th} CFS is $\mathbf{x}_j = [x_{1_j}^T\cdots x_{n_j}^T]^T \in \mathbb{R}^R$ (respectively, $\mathbf{y}_j = [y_{1_j}\cdots y_{n_j}]^T \in \mathbb{R}^n$). The vector of the i^{th} output from each CFS is denoted by $Y_i = [y_{i_1}\cdots y_{i_N}]^T \in \mathbb{R}^N$. The vector of all the states (respectively, outputs) is $X = [\mathbf{x}_1^T\cdots\mathbf{x}_N^T]^T$ (respectively, $Y = [\mathbf{y}_1^T\cdots\mathbf{y}_N^T]^T$). The stacked vector of external inputs to each CFS is denoted by U_{ext}, i.e., $U_{\text{ext}} = [u_{\text{ext}_1}\cdots u_{\text{ext}_N}]^T \in \mathbb{R}^N$. The Δ operator represents the difference between two distinct solutions of a system and between the inputs and the outputs associated with the two solutions. For example, $\Delta y_i = y_{i_a} - y_{i_b}$, where y_{i_a} and y_{i_b} are corresponding outputs for the ith CFS resulting from two solutions, x_{i_a} and x_{i_b}. $I_N \in \mathbb{R}^{N\times N}$ is the identity matrix and $\mathbf{1}_\ell(\mathbf{0}_\ell)$ is a vector of ones (zeros) in $\mathbb{R}^\ell$ for $\ell \in \mathbb{N}$.

D. Incremental Output Feedback Passivity of CFS

We now establish conditions under which CFS are iOFP.

Theorem 1: Consider the CFS depicted in Fig. 1, given by (3), (4), (5). If each block H_i, $i = 1,\cdots,n$ is iOSP with an incremental secant gain γ_i then the CFS is iOFP($-k$) with $k > ((-1+\gamma_1\cdots\gamma_n(\cos(\pi/n))^n)/\gamma_1)$.

Proof: Assuming that all the blocks H_i are iOSP with an incremental secant gain γ_i, there exist radially unbounded incremental storage functions $V_i(\Delta x_i) > 0 \forall \Delta x_i \neq 0$, $V_i(0) = 0$, $i = 1,\cdots,n$ such that $\dot{V}_1 \le -(\Delta y_1)^2 + \gamma_1\Delta y_1(\Delta u_{\text{ext}} - \Delta y_n)$ and

$$\dot{V}_i \le -(\Delta y_i)^2 + \gamma_i\Delta y_i\Delta y_{i-1} \tag{9}$$

for all $i = 2,\cdots,n$. Scaling V_1 by $1/(1+k\gamma_1)$, $k > 0$, we obtain $\dot{V}_1 \le -(1/(1+k\gamma_1))(\Delta y_1)^2 + (\gamma_1/(1+k\gamma_1))\Delta y_1(\Delta u_{\text{ext}} - \Delta y_n)$. Then, adding and subtracting $k\gamma_1/(1+k\gamma_1)(\Delta y_1)^2$, and defining $\tilde{\gamma}_1 = \gamma_1/(1+k\gamma_1)$, we can write

$$\dot{V}_1 \le -(\Delta y_1)^2 - \tilde{\gamma}_1\Delta y_1\Delta y_n + k\tilde{\gamma}_1(\Delta y_1)^2 + \tilde{\gamma}_1\Delta y_1\Delta u_{\text{ext}} \tag{10}$$

Combining (10) with (9), gives the incremental storage function for the entire CFS, $\mathbf{V} = \sum_{i=1}^n d_iV_i$ with $d_i > 0$, $i = 1,\cdots,n$, which satisfies the incremental dissipation inequality $\dot{\mathbf{V}} \le \Delta\mathbf{y}^T(\tilde{A}_k^TD + D\tilde{A}_k)\Delta\mathbf{y} + k\tilde{\gamma}_1(\Delta y_1)^2 + \tilde{\gamma}_1\Delta y_1\Delta u_{\text{ext}}$ where $\Delta\mathbf{y} = [\Delta y_1\cdots\Delta y_n]^T$, $D = \text{diag}\{d_1,\cdots,d_n\}$ and

$$\tilde{A}_k = \begin{pmatrix} -1 & 0 & \dots & 0 & -\tilde{\gamma}_1 \\ \gamma_2 & -1 & 0 & \dots & 0 \\ 0 & \gamma_3 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_n & -1 \end{pmatrix}.$$

From [18], we know that if

$$\tilde{\gamma}_1\gamma_2\cdots\gamma_n < \sec^n\left(\frac{\pi}{n}\right) \tag{11}$$

then positive scalars d_i, $i = 1,\cdots,n$ can be chosen so that the Lyapunov inequality $\tilde{A}_k^TD + D\tilde{A}_k \le -\epsilon_kI_n$ is satisfied for some $\epsilon_k = \epsilon(k) > 0$. Using the definition $\tilde{\gamma}_1 = \gamma_1/(1+k\gamma_1)$, we see that (11) can always be met by choosing k sufficiently large. In particular, for $k > ((-1+\gamma_1\cdots\gamma_n(\cos(\pi/n))^n)/\gamma_1)$ we obtain

$$\dot{\mathbf{V}} \le -\epsilon_k(\Delta\mathbf{y})^T(\Delta\mathbf{y}) + \tilde{\gamma}_1\left(k(\Delta y_1)^2 + \Delta y_1\Delta u_{\text{ext}}\right),\ \epsilon_k > 0 \tag{12}$$

showing that the CFS with input u_{ext} and output y_1 is iOFP($-k$) with $k > ((-1+\gamma_1\cdots\gamma_n(\cos(\pi/n))^n)/\gamma_1)$. ■

IV. State Synchronization in CFS Networks

Consider a network composed of N identical CFS of the form (3), (4), (5), with iOSP blocks H_i. Given any two sets of initial conditions, inputs, states and outputs for two CFS j,m, the incremental storage function $\mathbf{V}_{j,m}$ satisfies an incremental dissipation inequality of the form (12). Defining $\Delta x_{i_{j,m}} = x_{i_j} - x_{i_m}$ and $\Delta\mathbf{x}_{j,m} = (\Delta x_{1_{j,m}}^T,\cdots,\Delta x_{n_{j,m}}^T)^T$, $\mathbf{V}_{j,m}$ satisfies $\mathbf{V}_{j,m}(\Delta\mathbf{x}_{j,m}) > 0\ \forall\Delta\mathbf{x}_{j,m} \neq \mathbf{0}_R$, $\mathbf{V}_{j,m}(\mathbf{0}_R) = 0$ and the incremental dissipation inequality

$$\begin{aligned}\dot{\mathbf{V}}_{j,m} \le & -\epsilon_k(\Delta\mathbf{y}_{j,m})^T(\Delta\mathbf{y}_{j,m}) \\ & + \tilde{\gamma}_1\left(k\left(\Delta y_{1_{j,m}}\right)^2 + \Delta y_{1_{j,m}}\Delta u_{\text{ext}_{j,m}}\right)\end{aligned} \tag{13}$$

where $\Delta y_{i_{j,m}} = y_{i_j} - y_{i_m}$, $\Delta\mathbf{y}_{j,m} = [\Delta y_{1_{j,m}}\cdots\Delta y_{n_{j,m}}]^T$, $\epsilon_k > 0$. From Theorem 1, a CFS with iOSP blocks, each with incremental secant gain γ_i, satisfies (13) provided that $k > ((-1+\gamma_1\cdots\gamma_n(\cos(\pi/n))^n)/\gamma_1)$.

We also define the following property which we will assume of the isolated CFS. This assumption will be used to deduce state synchronization from output synchronization:

Definition 1 Limit Set Detectability: Let $\psi_{\mathbf{x}_j} \subset \mathbb{R}^R$ (respectively, $\psi_{\mathbf{y}_j} \subset \mathbb{R}^n$) represent the invariant state (respectively, output) omega-limit set of autonomous CFS j. Consider $\psi_{\mathbf{y}_j}, \psi_{\mathbf{y}_m}$, where $j \neq m$ and $j,m \in \{1,\cdots,N\}$. The CFS is limit set detectable iff $\psi_{\mathbf{y}_j} = \psi_{\mathbf{y}_m}$ implies that $\psi_{\mathbf{x}_j} = \psi_{\mathbf{x}_m}$.

Remark 3: Note that CFS composed of blocks H_i of the form (6), (7) are limit-set detectable since $g_i(\cdot)$ is invertible.

A. Network Input-Output Coupling Rules

We assume that the CFS are connected through a weighted directed graph $\mathcal{G} = \{\mathcal{A},\mathcal{D}\}$ and restrict the coupling structure to a linear, static input-output interconnection where the j^{th} CFS is coupled to other

CFS nodes in the network through its input u_{ext_j} and output y_{1_j} using the Laplacian coupling matrix $\Gamma \in \mathbb{R}^{N\times N}$ so that $U_{\mathrm{ext}} = -\Gamma Y_1$.

Definition 2 Weighted Adjacency Matrix: A matrix $\mathcal{A} = \{w_{j,l}\}$, $j,l = 1,\cdots,N$, $\mathcal{A} \in \mathbb{R}^{N\times N}$, where $w_{j,l}$ represents the weight of the edge from node l to node j. We assume that the graph is simple, i.e. $w_{j,l} \geq 0$ and $w_{j,j} = 0, \forall j,l$.

Definition 3 Degree Matrix: A matrix $\mathcal{D} = \mathrm{diag}\{\delta_j\}$, $j = 1,\cdots,N$, associated with $\mathcal{A}$.

Definition 4 Laplacian Matrix: A matrix $\Gamma = \mathcal{D} - \mathcal{A} = \{\Gamma_{j,l}\}$, $j,l = 1,\cdots,N$ associated with $\mathcal{A}$, with $\Gamma_{j,j} = \sum_{l\neq j}^{N} w_{j,l}, \forall j = 1,\cdots,N$ and $\Gamma_{j,l} = -w_{j,l}, \forall j \neq l$.

The interconnection rule $U_{\mathrm{ext}} = -\Gamma Y_1$ then corresponds to the linear consensus protocol $u_{\mathrm{ext}_j} = -\sum_{l=1}^{N} w_{j,l}(y_{1_j} - y_{1_l})$ (see [23]). We make the following assumptions on Γ:

$$\begin{aligned}&(\mathbf{A1})\ \mathrm{rank}\,(\Gamma) = N-1\\ &(\mathbf{A2})\,\Gamma + \Gamma^T \geq 0\\ &(\mathbf{A3})\,\Gamma\mathbf{1}_N = \Gamma^T\mathbf{1}_N = \mathbf{0}_N.\end{aligned}$$

Assumption (A1) holds provided that the graph is strongly connected and simple [23]. Assumption (A3) holds if the graph is balanced, i.e. if $\mathcal{A}\mathbf{1}_N = \mathcal{A}^T\mathbf{1}_N$, and it implies (A2) [24].

B. State Synchronization in Networks of Identical CFS

This section gives the main result on global asymptotic state synchronization. In Theorem 2, $\lambda_2 = \lambda_2(\Gamma_s)$ denotes the second smallest eigenvalue of $\Gamma_s = 1/2(\Gamma + \Gamma^T)$.

Theorem 2: Consider a network of N identical CFS of the form (3), (4), (5), linearly coupled through the Laplacian Γ, i.e. $U_{\mathrm{ext}} = -\Gamma Y_1$ where Γ satisfies (A1)–(A3). Assume that each CFS is limit set detectable as per Definition 1 and is iOFP$(-k)$ so that for every CFS pair $j,m \in \{1,\cdots,N\}$ there exists a radially unbounded incremental storage function $\mathbf{V}_{j,m}$ satisfying (13), and that the network satisfies the strong coupling assumption $\lambda_2(\Gamma_s) \geq k$. Then, each network solution that exists for all $t \geq 0$ is such that $\forall i = 1,\cdots,n, \forall j,l = 1,\cdots,N$: $\lim_{t\to+\infty}(x_{i_j}(t) - x_{i_l}(t)) = \mathbf{0}_{r_i}$. In addition, any bounded network solution is such that the state solution of each CFS converges to the omega-limit set of the isolated CFS.

Proof: To compare each CFS output with its average over all the N CFS outputs we employ the projector $\Pi = I_N - (1/N)\mathbf{1}_N\mathbf{1}_N^T$. This projector measures the instantaneous difference between a signal and its average over all CFS in the network, e.g. the j^{th} element of $\Pi Y_1(t)$ measures the difference between output $y_{1_j}(t)$, $j = 1,\cdots,N$ and the average output $(1/N)\sum_{j=1}^{N} y_{1_j}(t)$. Summing storage functions $\mathbf{V}_{j,m}$ in (13) for all CFS pairs j,m and scaling by $1/2N$ gives the incremental storage function $S(X) = (1/2N)\sum_{j=1}^{N}\sum_{m=1}^{N}\mathbf{V}_{j,m}$ for the network. Using (13), S obeys the dissipation inequality

$$\begin{aligned}\dot{S} \leq& -\epsilon_k\left((\Pi\otimes I_n)Y\right)^T\left((\Pi\otimes I_n)Y\right)\\ &+\tilde{\gamma}_1\left(k(\Pi Y_1)^T\Pi Y_1 + (\Pi Y_1)^T\Pi U_{\mathrm{ext}}\right).\end{aligned}\qquad(14)$$

Since $U_{\mathrm{ext}} = -\Gamma Y_1$ and (A3), we have $\Pi U_{\mathrm{ext}} = -\Pi\Gamma Y_1 = -\Gamma\Pi Y_1$ so that (14) can be rewritten as

$$\begin{aligned}\dot{S} \leq& -\epsilon_k\left((\Pi\otimes I_n)Y\right)^T\left((\Pi\otimes I_n)Y\right)\\ &+\tilde{\gamma}_1\left(k(\Pi Y_1)^T\Pi Y_1 - (\Pi Y_1)^T\Gamma\Pi Y_1\right).\end{aligned}\qquad(15)$$

Using (A1)–(A3), we have $\Pi Y_1 = Y_1 - (1/N)\mathbf{1}_N^T Y_1\mathbf{1}_N = \mathbf{0}_N$ iff $Y_1 \in \ker(\Gamma)$ (see [11]) and

$$(\Pi Y_1)^T\Gamma\Pi Y_1 \geq \lambda_2(\Gamma_s)(\Pi Y_1)^T\Pi Y_1 \qquad(16)$$

Using (16) in (15), we obtain

$$\dot{S}\leq -\epsilon_k((\Pi\otimes I_n)Y)^T((\Pi\otimes I_n)Y) + \tilde{\gamma}_1(k-\lambda_2(\Gamma_s))(\Pi Y_1)^T\Pi Y_1$$

which, if $\lambda_2(\Gamma_s) \geq k$ (strong coupling), yields

$$\dot{S} \leq -\epsilon_k\left((\Pi\otimes I_n)Y\right)^T\left((\Pi\otimes I_n)Y\right). \qquad(17)$$

Letting $S_0 = S(X(0))$, the initial value of the incremental storage function for the whole network, we note that, since $S \geq 0$ and $\dot{S} \leq 0$, the set $\mathcal{M} = \{X | S(X) \leq S_0\}$ is an invariant set. From (17), and using the LaSalle invariance principle, the incremental signal $(\Pi\otimes I_R)X$ will converge to the largest invariant subset included in $\{X|\dot{S} = 0\} \cap \mathcal{M}$ as $t \to \infty$. Due to (17), $\dot{S} = 0$ only if $(\Pi\otimes I_n)Y = \mathbf{0}_{Nn}$. Convergence to a set wherein $(\Pi\otimes I_n)Y = \mathbf{0}_{Nn}$ implies asymptotic output synchronization since $\forall i$, and for any $j,l \in \{1,\cdots,N\}$: $\lim_{t\to+\infty}(y_{i_j}(t) - y_{i_l}(t)) = 0$. Under the limit set detectability assumption, if any two isolated CFS j,m have identical asymptotic output behaviors ($\lim_{t\to\infty}(\mathbf{y}_m - \mathbf{y}_j) = \mathbf{0}_n$), their states must also be identical asymptotically ($\lim_{t\to\infty}(\mathbf{x}_m - \mathbf{x}_j) = \mathbf{0}_R$), implying asymptotic state synchronization. Since $\Gamma\mathbf{1}_N = \mathbf{0}_N$, the effect of the coupling disappears when output synchrony is reached and each CFS in the network is then effectively isolated. Therefore, for any bounded network solution, the solution of each CFS converges to the omega-limit set of an isolated CFS. ■

From Theorems 1 and 2, we see that a sufficient condition for the asymptotic synchronization of linearly interconnected, identical CFS is $\lambda_2(\Gamma_s) \geq k > ((-1 + \gamma_1\cdots\gamma_n(\cos(\pi/n))^n)/\gamma_1)$.

C. Comparison With Other Synchronization Results

Previous approaches to the synchrony problem include the LMI approach of [25], which considers a quadratic Lyapunov function operating on the difference between corresponding nodal states. If such a Lyapunov function can be found such that its time derivative, cast as an LMI, is negative definite, then synchrony is guaranteed. Our incremental dissipativity approach simplifies the construction of the Lyapunov function, but limits the class of Lyapunov functions to the linear sum of the incremental storage functions of nodal subsystems.

Following [26], which this present technical note extends, [27] used an input-output approach to derive synchronization conditions for general nodes composed of the interconnection of subsystems which have an iOFP property. As with [26] and this paper, [27] constructs an interconnection matrix from the structure of the nodes, parameterized by the degree of incremental passivity of each nodal subsystem. But whereas [27] uses input-output arguments to show that diagonal stability of the interconnection matrix leads to the $\mathcal{L}_2$ stability of the incremental output, this technical note approaches the synchrony problem in two steps: first, a virtual static feedback of gain k is placed around the coupling subsystem, to reduce its incremental secant gain (Fig. 2). Theorem 1 shows that the feedback strength k that sufficiently reduces the incremental secant gain $\tilde{\gamma}_1$ to the extent that the interconnection matrix $\tilde{A}_k$ is diagonally stable quantifies the shortage of incremental passivity of the CFS. Second, Theorem 2 shows that strong network coupling acts as an incremental output feedback that compensates for the shortage of incremental passivity k. For the case of CFS, this technical note and [27] both use the secant gain condition to derive the coupling strength required for diagonal stability of the interconnection matrix, and therefore the final condition on network coupling strength is the same. Regarding the synchrony problem from this angle presents network coupling as an analogue to a stabilizing feedback. This gives synchrony an intuitive interpretation, and suggests the use of classical feedback stabilization methods for studying the synchronization problem, as recently done in [28].

The partial contraction approach of [7] provides a sufficient condition for the synchronization of coupled nodes, namely that a matrix measure of the Jacobian of an auxiliary system (which has as particular solutions the solutions of the individual coupled nodes) be negative definite. To link this technical note's methodology to [7], consider a network of N CFS of the form (6), (7), (5), coupled via outputs $y_{1_j} = g_1(x_{1_j})$ using the Laplacian Γ of the all-to-all case given in Section V. With this coupling, the external input to the j^{th} CFS is $u_{\text{ext}_j} = -w\sum_{l=1}^{N}(y_{1_j} - y_{1_l}) = -Nwy_{1_j} + \sum_{l=1}^{N} y_{1_l}$ and $\lambda_2(\Gamma_s) = Nw$. Let $\tilde{\mathbf{x}} = [\tilde{x}_1 \cdots \tilde{x}_n]^T \in \mathbb{R}^n$ be the state vector of the auxiliary system in [7], given by

$$\begin{aligned}\dot{\tilde{x}}_1 &= -g_n(\tilde{x}_n) - f_1(\tilde{x}_1) - Nwg_1(\tilde{x}_1) + \sum_{l=1}^{N} g_1(x_{1_l})\\ \dot{\tilde{x}}_2 &= g_1(\tilde{x}_1) - f_2(\tilde{x}_2)\\ &\vdots\\ \dot{\tilde{x}}_n &= g_{n-1}(\tilde{x}_{n-1}) - f_n(\tilde{x}_n). \end{aligned} \tag{18}$$

Partial contraction requires that (18) is contracting with respect to $\tilde{\mathbf{x}}$. Jacobian $A(\tilde{\mathbf{x}}) \in \mathbb{R}^{n\times n}$ of (18) has the structure

$$A(\tilde{\mathbf{x}}) = \begin{bmatrix} -a_1(\tilde{x}_1) & 0 & \cdots & 0 & -b_n(\tilde{x}_n)\\ b_1(\tilde{x}_1) & -a_2(\tilde{x}_2) & 0 & \cdots & 0\\ 0 & b_2(\tilde{x}_2) & -a_3(\tilde{x}_3) & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & b_{n-1}(\tilde{x}_{n-1}) & -a_n(\tilde{x}_n) \end{bmatrix} \tag{19}$$

and is such that $a_1(\tilde{x}_1) = f_1'(\tilde{x}_1) + Nwg_1'(\tilde{x}_1)$, $a_i(\tilde{x}_i) = f_i'(\tilde{x}_i)$, $(i = 2,\cdots,n)$, $b_i(\tilde{x}_i) = g_i'(\tilde{x}_i)$ $(i = 1,\cdots,n)$. Note that $A(\tilde{\mathbf{x}})$ has the same structure as the matrix $\tilde{A}_k$ in the proof of Theorem 1. Such matrices are Hurwitz if $(b_1(\tilde{x}_1)\cdots b_n(\tilde{x}_n)/a_1(\tilde{x}_1)\cdots a_n(\tilde{x}_n)) < \sec^n(\pi/n)$ [14], in which case there exists a symmetric matrix $Q(\tilde{\mathbf{x}}) > 0$ such that $A(\tilde{\mathbf{x}})^T Q(\tilde{\mathbf{x}}) + Q(\tilde{\mathbf{x}})A(\tilde{\mathbf{x}}) < 0$.

If the conditions of Theorem 2 are satisfied with $\lambda_2(\Gamma_s) = Nw \geq k$ and $k > ((-1 + \gamma_1\cdots\gamma_n(\cos(\pi/n))^n)/\gamma_1)$, then assuming that $\sup_{\tilde{x}_i}(g_i'(\tilde{x}_i)/f_i'(\tilde{x}_i))$ exists for all i and defining $\gamma_i = \sup_{\tilde{x}_i}(g_i'(\tilde{x}_i)/f_i'(\tilde{x}_i)\forall i)$ and $\tilde{\gamma}_1 = \gamma_1/(1 + \lambda_2(\Gamma_s))\gamma_1 = \gamma_1/(1 + Nw\gamma_1)$, it follows that $\tilde{\gamma}_1 = \sup_{\tilde{x}_1}(b_1(\tilde{x}_1)/a_1(\tilde{x}_1))$ and $\tilde{\gamma}_1\gamma_2\cdots\gamma_n = \Pi_{i=1}^{n}\sup_{\tilde{x}_i}(b_i(\tilde{x}_i)/a_i(\tilde{x}_i)) < \sec^n(\pi/n)$. Therefore if the conditions of Theorem 2 are met

1) incremental dissipativity can be used to construct an incremental storage function from the incremental storage functions of the CFS subsystems (as in Theorem 2);
2) the Jacobian $A(\tilde{\mathbf{x}})$ is Hurwitz $\forall\tilde{\mathbf{x}} \in \mathbb{R}^n$ and $\exists Q(\tilde{\mathbf{x}}) = Q(\tilde{\mathbf{x}})^T > 0$ so that $A(\tilde{\mathbf{x}})^T Q(\tilde{\mathbf{x}}) + Q(\tilde{\mathbf{x}})A(\tilde{\mathbf{x}}) < 0$. Matrix $Q(\tilde{\mathbf{x}}) = \text{diag}(q_1(\tilde{\mathbf{x}}),\cdots,q_n(\tilde{\mathbf{x}}))$ can be constructed by modifying the diagonal matrix D in [18] to: $q_1(\tilde{\mathbf{x}}) = a_1(\tilde{x}_1)$ and $q_i(\tilde{\mathbf{x}}) = \frac{a_i(\tilde{x}_i)p(\tilde{\mathbf{x}})^{2(i-1)}}{(\frac{b_1(\tilde{x}_1)\cdots b_{i-1}(\tilde{x}_{i-1})}{a_1(\tilde{x}_1)\cdots a_{i-1}(\tilde{x}_{i-1})})^2}$ for $i = 2,\cdots,n$, where $p(\tilde{\mathbf{x}}) = (\frac{b_1(\tilde{x}_1)\cdots b_n(\tilde{x}_n)}{a_1(\tilde{x}_1)\cdots a_n(\tilde{x}_n)})^{1/n}$.

Note that if subsystems H_i are not first order as in (6), (7), the incremental passivity characterization of H_i allows the use of Theorems 1 and 2 to prove synchrony without a state-space analysis of the auxiliary system's Jacobian, which can become analytically intractable if blocks H_i are of high dimension.

The algorithm of [8], [29] gives sufficient conditions for the Gershgorin disks of the Jacobian of the auxiliary system of [7] to reside in the left half of the complex plane. Contraction can be proved using the algorithm of [8] by verifying the negativity of a matrix measure of the auxiliary system's Jacobian (e.g. the 1-, 2- and ∞-matrix measures). Tight Gershgorin disks may be obtained by scaling the Jacobian by a similarity transformation. However, there is no systematic way of constructing such scalings, without which the Gershgorin radius can be conservative. Furthermore, whilst contraction relies on a full internal description of network nodes, the method of this technical note relies only on the incremental dissipativity properties of the subsystems, encompassed in gains γ_i, and thus little knowledge of the nodal dynamics is needed.

Also note that for CFS networks of general topologies, $A(\mathbf{x})$ is identical to the variational equations of the master stability function (MSF) approach of [2] except that the expression of $a_1(x_1)$ is replaced by $a_1(x_1, m) = f_1'(x_1) + \lambda_m(\Gamma_s)g_1'(x_1)$, where $m = 2,\cdots,N$ and $\lambda_m(\Gamma_s)$ are the eigenvalues of Γ_s, the symmetric part of Γ (note that, from (A1) and (A2), $0 < \lambda_2(\Gamma_s) \leq \cdots \leq \lambda_N(\Gamma_s)$). If the conditions of Theorem 2 are satisfied with $\lambda_2(\Gamma_s) \geq k > ((-1 + \gamma_1\cdots\gamma_n(\cos(\pi/n))^n)/\gamma_1)$ and if $\gamma_i \triangleq \sup_{x_i}(g_i'(x_i)/f_i'(x_i))\forall i$, $\tilde{\gamma}_1 = \gamma_1/(1 + \lambda_2)(\Gamma_s)\gamma_1$, then $\tilde{\gamma}_1 = \sup_{x_1}(b_1(x_1)/a_1(x_1, 2)) \geq (b_1(x_1)/a_1(x_1, m))$, $\forall m$, and by a similar argument to the above, the variational equations of [2] are Hurwitz for all m and the synchronous solution is stable. The converse is not generally true as the MSF approach is a local analysis tool and the synchronous solution may only be locally stable.

V. Illustration: Network of Interconnected

Consider a network of N coupled biochemical oscillators, given in [16], each decomposable into four iOSP subsystems

$$\begin{aligned} &H_{1_j}: \dot{x}_{1_j} = -p_1 x_{1_j} + u_{1_j}, && y_{1_j} = p_1 x_{1_j}, \quad u_{1_j} = -y_{4_j}\\ &H_{2_j}: \dot{x}_{2_j} = -p_2 x_{2_j} + u_{2_j}, && y_{2_j} = p_2 x_{2_j}, \quad u_{2_j} = y_{1_j}\\ &H_{3_j}: \dot{x}_{3_j} = -f_3(x_{3_j}) + u_{3_j} + u_{\text{ext}_j}, && y_{3_j} = x_{3_j}, \quad u_{3_j} = y_{2_j}\\ &H_{4_j}: y_{4_j} = g_3(u_{4_j}), && u_{4_j} = y_{3_j} \end{aligned}$$

For $j = 1,\cdots,N$, $f_3(x_{3_j}) = x_{3_j}/(1+x_{3_j})$, $g_3(x_{3_j}) = -(10/(1+x_{3_j}))$. From [16], the model parameters are $p_1 = p_2 = 0.01$. The CFS are linearly interconnected using their external input u_{ext_j} and output y_{3_j} via Laplacian $\Gamma \in \mathbb{R}^{N\times N}$, which satisfies (A1), (A2), and (A3), thus: $U_{\text{ext}} = -\Gamma Y_3$. With coupling gain w, Γ has the following structure for the two topologies we consider:

$All-to-all$:

$$\Gamma = Nw\left(I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right)$$

$$\lambda_2 = Nw$$

Unidirectional ring:

$$\Gamma = \begin{bmatrix} w & -w & \cdots & 0 & 0\\ 0 & w & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & w & -w\\ -w & 0 & \cdots & 0 & w \end{bmatrix}$$

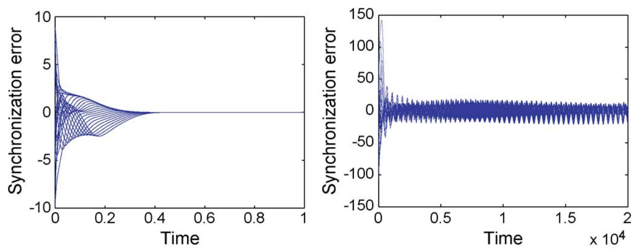
$$\lambda_2 = \left(1 - \cos\left(\frac{2\pi}{N}\right)\right)w.$$

Note that the positive orthant of the coupled CFS is an invariant set [26], and we can therefore limit the analysis to this set assuming the initial conditions lie therein. Each of H_1, H_2 and H_4 are iOSP with incremental secant gains $\gamma_1 = \gamma_2 = 1$ and $\gamma_4 = 10$ respectively. Subsystem H_3 has infinite incremental secant gain γ_3 since the $\inf_{x_3\geq 0} f_3'(x_3) = 0$.

From Theorem 2, if each node is iOFP$(-k)$ and if the input/output pair u_{ext_j}/y_{3_j} is a coupling port such that $U_{\text{ext}} = -\Gamma Y_3$, then the CFS states will asymptotically synchronize if $\lambda_2 \geq k$. From Theorem 1, the condition for each node to be iOFP$(-k)$ is $k > ((-1 + \gamma_1\cdots\gamma_4\cos^4(\pi/4))/\gamma_3) = \gamma_1\gamma_2\gamma_4\cos^4(\pi/4) = 10\cos^4(\pi/4) = 2.5$, and therefore the synchronization condition is $\lambda_2 \geq k > 2.5$. The minimum values $w_{\min}$ of the coupling gain w needed to achieve the required λ_2 are shown in Table I for each topology with $N = 4,25$.

TABLE I
COUPLING STRENGTHS w SATISFYING THEOREM 2

		Topology	
		All-to-all	Unidirectional ring
No. of CFS	$N = 4$	$w_{\min} = 0.625$	$w_{\min} = 2.5$
	$N = 25$	$w_{\min} = 0.025$	$w_{\min} = 79.6$



(a) (b)

Fig. 3. Synchronization error of y_{3_j} (given by $y_{3_j} - (1/N)\sum_{k=1}^{N} y_{3_k}$), for 25 CFS coupled in a unidirectional ring. (a) Coupling gain $w = 80$; (b) coupling gain $w = 0.01$.

The synchronization error of the outputs y_{3_j} (given by $y_{3_j} - (1/N)\sum_{\kappa=1}^{N} y_{3_\kappa}$) for a network of 25 CFS interconnected in a unidirectional ring topology $(j = 1, \cdots, 25)$ with coupling strength $w = 80$ is shown in Fig. 3(a). Simulation results show that global synchronization is not guaranteed when $\lambda_2 < k$, as shown in Fig. 3(b), where $w = 0.01$. The fact that at the low coupling strength $w = 0.01$ synchronization does not take place demonstrates that synchrony requires that w be above a certain threshold. However, the lower bound on the coupling strength obtained using Theorem 2 is conservative $(w_{\min=79.6})$. A source of conservativeness is the globality of the result. However, what we gain at the price of conservativeness is robustness to model variations: the incremental dissipation inequality of a subsystem with an incremental secant γ_i gain can be used to represent the (wide) class of iOSP systems that have an excess of incremental passivity of $1/\gamma_i$. Fixing the coupling strength whilst replacing a subsystem H_i with one having equal or smaller γ_i would not affect synchrony. In the biological setting, where parameters typically vary significantly, placing a plausible upper bound on the quantity γ_i therefore allows us to analyze synchrony in a way that is robust to such parametric variations.

VI. DISCUSSION

We have presented sufficient conditions for state synchronization in networks of CFS. The method relies upon quantifying the nodes' degree of iOFP and showing that strong coupling can compensate for any shortage of incremental passivity, rendering the interconnected nodes iOSP. With a limit set detectability assumption, this leads to global asymptotic state synchronization in CFS networks. In contrast to other methodologies, this approach is constructive and requires minimal knowledge of the nodal dynamics, making it applicable to network synchronization analysis and design problems.

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