Problem Sheet 2: Eigenvalues and eigenvectors and their use in solving linear ODEs

If you find any typos/errors in this problem sheet please email jk208@ic.ac.uk.

The material in this problem sheet is not examinable. It is intended for the more mathematically-inclined students who want to obtain a more thorough understanding of eigenvalues and eigenvectors and their use in solving linear ODEs. Even though in the course we only deal with models of dimension 1 and 2, here we consider vectors, matrices and linear models of arbitrary (but finite) dimension n. The reason why is because we believe that, when discussing the material in this problem sheet, nothing is gained in terms of simplicity and comprehensibility, quite the contrary, by limiting the dimensions of the vector/matrices we deal with to be of a fixed low dimension. However, any course assessment will only require you to be able to apply the concepts discussed below to vectors, matrices and linear models of dimension 1 or 2.

1. An *eigenvector* v of an $n \times n$ matrix of real numbers A is defined as a non-trivial vector[1] of complex numbers such that $Av = \lambda v$ where $\lambda \neq 0$ is a complex number called an *eigenvalue*. If real, eigenvalues and eigenvectors have very simple geometric interpretations. For instance, an eigenvector v is a vector such that A maps it onto itself, in other words, Av lies on the same line as v, scaled (up if $|\lambda| > 1$ or down if $|\lambda| < 1$) and/or rotated by 180^o (if $\lambda < 0$). Further information on linear algebra can be found in

   - `http://www.khanacademy.org/math/linear-algebra` (short camcasts which are excellent if you have never seen any linear algebra before in your life).
   - `http://see.stanford.edu/see/lecturelist.aspx?coll=17005383-19c6-49ed-9497-2ba8bfcfe5f6` (bit more advanced, great lecturer, first 7 lectures cover useful linear algebra concepts with engineering-type applications).
   - Linear algebra done right by Sheldon Axler (it is a nice, brief and rigorous introduction to the subject that doesn't rely on determinants for proofs, which in our opinion is a good thing).
   - Any other of the numerous textbooks on the subject.

   (a) We can represent any vector $x \in \mathbb{R}^n$ as a linear combination of the eigenvectors of A if and only if A has n eigenvectors that form a linearly independent set vectors[2]. In the words, if the previous is true, and letting $v_1, v_2, \dots, v_n$ denote n linearly independent eigenvectors, then for any given $x \in \mathbb{R}^n$ we can find some complex numbers $c_1, c_2, \dots, c_n$ such that

   $$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

   Use the above to find an expression for the vector $c = [c_1, c_2, \dots, c_n]^T$ in terms of the matrix of linearly independent eigenvectors $V = [v_1, v_2, \dots, v_n]$ and in terms of x. (Note that $v_1, v_2, \dots, v_n$ are column vectors corresponding to the 1st, 2nd, ..., and nth column of the matrix V, respectively.) *Hint: The inverse of a matrix exists if and only if its columns form a linearly independent set of vectors. In addition, if it exists it is unique.*

   (b) Eigenvectors and eigenvalues are useful because they provide us a very straightforward way of determining how A transforms any given vector x (i.e., what Ax looks like). Show that

   $$Ax = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_n \lambda_n v_n,$$

   where λ_i denotes the eigenvalue such that $Av_i = \lambda_i v_i$ with $i \in \{1, 2, \dots, n\}$. λ_i is said to be the eigenvalue *corresponding to*, or *associated with*, eigenvector v_i.

[1] A vector is said to be *non-trivial* if at least one of its elements is non-zero.

[2] A set of vectors $\{v_1, v_2, \dots, v_n\}$ is said to be linearly independent if none of the vectors is a linear combination of the other vectors, for example $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\right\}$ are linearly independent, while $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}4\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ are not (they are said to be *linearly dependent* sets of vectors).

(c) Using the answers to the previous two parts, show that $Ax = V\Lambda V^{-1}x$, where Λ is an $n \times n$ diagonal matrix such that $\Lambda_{ii} = \lambda_i$. Expressing A as $V\Lambda V^{-1}$ is called *diagonalising* A. If it is possible to diagonalise A, then A is said to be*diagonalisable*. It is possible to do this if and only if one can find n eigenvectors that form a set of linearly independent vectors. Otherwise we can't carry out the step we did in part (a); we can't express any arbitrary $x \in \mathbb{R}^n$ as a linear combination of the eigenvectors of A.

(d) All the above is very well, but to put it to practice one must first be able to find the eigenvectors and eigenvalues of a given matrix A. A property of determinants is that, for any given matrix A, there exists a non-trivial vector x such $Ax = 0$ if and only if $\det(A) = 0$. By the definition of eigenvectors, $Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow (A - I\lambda)v = 0 \Leftrightarrow \det(A - I\lambda) = 0$ for any eigenvalue λ. Note that $\det(A - I\lambda)$ is a polynomial in λ, also note that this implies that there is at most n eigenvalues (can you see why?, *Hint: think about the order of the polynomial* $\det(A - I\lambda)$). Thus, by solving for the roots of $\det(A - I\lambda)$ we find the eigenvalues. Practice this by finding the eigenvalues of the following matrices

$$i) \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}, \quad ii) \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad iii) \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad iv) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(e) Once we have the eigenvalues then we simply use $(A - \lambda I)v = 0$ to figure out the eigenvectors. For the above four matrices find all linearly independent eigenvectors[3].

(f) Which of the four matrices are diagonalisable?

2. In this exercise we derive the solution of arbitrary (but finite) dimensional linear models defined by

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{1}$$

where A is $n \times n$ and diagonalisable. In other words we find a function $x(t)$ that obeys two rules: **I.** $\dot{x}(t) = Ax(t)$ for all $t \geq 0$, i.e., that at any given time, it's time-derivative equals A times itself and **II.** $x(0) = x_0$, i.e., its value at time 0 is x_0. In addition, with no extra effort we show that there is only one such solution $x(t)$, i.e., that the solution $x(t)$ is **unique**[4].

(a) Show that $\frac{d}{dt}e^{\Lambda t} = \Lambda e^{\Lambda t} = e^{\Lambda t}\Lambda$, where Λ is as in exercise 1 and

$$e^{\Lambda t} := \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}.$$

Hint: If D_1 and D_2 are diagonal $n \times n$ matrices, then $D_1 D_2 = D_2 D_1$.

(b) Pre-multiply $\dot{x}(t) = Ax(t)$ by $(Ve^{-\Lambda t}V^{-1})$ and show that $x(t) = Ve^{\Lambda t}V^{-1}c$ where $c \in \mathbb{R}^n$ is a vector of constants. *Hint 1:* Remember integrating factors? *Hint 2:* If M, N and $Q(t)$ are $n \times n$ matrices such that $Q(t)$ varies in time but M, N do not, then $\frac{d}{dt}(MQ(t)N) = M\frac{d}{dt}(Q(t))N$. In addition, if $v(t) \in \mathbb{R}^n$ varies with time, then $\frac{d}{dt}(Q(t)v(t)) = \frac{d}{dt}(Q(t))v(t) + Q(t)\frac{d}{dt}(v(t))$[5].

(c) Show that $e^{\Lambda 0} = I$, where I is the identity matrix.

(d) Use part (c) and rule **II.** to show that $c = x_0$.

[3]This is 'slang', linear independence is a property of a sets of vectors not of vectors; to say that a vector v is linearly independent has no meaning. However, it is common practice to say that a bunch of vectors are linearly independent if the set of those vectors is linearly independent.

[4]It is well known that there only exists one solution, that is why we write "we will derive **the** solution to (1)" instead of "we will derive **a** solution to (1)". Indeed, generally, it is best to say "the unique solution" instead of "the solution" as it removes all possible ambiguity.

[5]This is one of the generalisations of the product rule in multivariable calculus.

(e) Is $x(t) = Ve^{\Lambda t}V^{-1}x_0$ the unique solution to (1)?

3. (More advanced) In this exercise we take a different approach to show that

$$x(t) = Ve^{\Lambda t}V^{-1}x_0, \tag{2}$$

as defined in exercise 2, is the unique solution to (1). In particular, we first show that, at most, (1) has a single solution. Then, separately, we show that (2) is **a** solution to (1) by verifying that it obeys rules **I.** and **II.** (see the description of exercise 2).

(a) The Gronwall-Bellman inequality implies that if a continuous function $y : [0, +\infty) \to \mathbb{R}$ satisfies

$$y(t) \leq \alpha + \int_0^t \beta y(\tau) d\tau$$

where $\alpha \in \mathbb{R}$ and $\beta \geq 0$ are constants, then

$$y(t) \leq \alpha e^{\beta t}. \tag{3}$$

Assume that the solution is not unique, i.e., that we have two functions, $x(t)$ and $z(t)$ both of which satisfy (1). Consider the norm difference between $x(t)$ and $z(t)$, i.e., $\delta(t) = ||x(t) - z(t)||$. Use (3) to show that $\delta \equiv 0$, that is that $\delta(t) = 0$ for all $t \geq 0$, and thus argue that the solution is indeed unique.

(b) Next, show that (2) satisfies rule **I.**, i.e., that $\dot{x}(t) = Ax(t)$. *Hint:* Use exercise $2(a)$ and the second hint given in exercise $2(b)$.

(c) Finish by showing that (2) satisfies rule **II.**, i.e., that $x(0) = x_0$. *Hint:* Use exercise $2(c)$.

Solutions

1. (a) The fact that the eigenvectors are linearly independent, in other words, that the columns of V are linearly independent, implies that the inverse of V exists, hence

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = Vc \Leftrightarrow c = V^{-1}x.$$

(b) This follows simply from linearity, $A(b_1 x_1 + b_2 x_2) = b_1 A x_1 + b_2 A x_2$ for any matrix A, vectors x_1, x_2 and scalars b_1, b_2.
From (a), we have that x, written in terms of the linearly independent eigenvectors of A, is given by $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$. Therefore, we have:

$$Ax = A(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \underset{c_i \text{ are scalars}}{=} c_1 A v_1 + c_2 A v_2 + \cdots + c_n A v_n = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_n \lambda_n v_n.$$

(c) Using (b) and (a), we have:

$$Ax \underset{\text{(using (b))}}{=} c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_n \lambda_n v_n = V \begin{bmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \vdots \\ \lambda_n c_n \end{bmatrix} = V \Lambda c \underset{\text{(using (a))}}{=} V \Lambda V^{-1} x.$$

(d) By the fundamental theorem of algebra, a polynomial of order n has at most n complex roots. Hence any matrix of dimension n has at most n eigenvalues.

i) $\lambda_1 = 2, \lambda_1 = 1$. A little trick is that the eigenvalues of triangular matrices are the elements on the main diagonal of the matrix.

ii) $\lambda_1 = 0, \lambda_2 = 3$.

iii) $\lambda_1 = 1, \lambda_2 = 1$ (repeated eigenvalue at 1).

iv) $\lambda_1 = 2, \lambda_2 = 2$ (repeated eigenvalue at 2).

(e) *i)* $v_1 = [1, 0]^T, v_2 = [1, 1]^T$, where v_1 denotes the eigenvector that corresponds to λ_1. Note that av_1 and av_2 are also valid eigenvectors for any non-zero real number a. However, the biggest set of linearly independent eigenvectors has two elements v_1 and v_2 or any scaled versions of them.

ii) $v_1 = [1, 1]^T, v_2 = [1, -1/2]^T$.

iii) $v_1 = [1, -1]^T$.

iv) $v_1 = [1, 0]^T$, $v_2 = [0, 1]^T$. There is one eigenvalue but two eigenvectors! This is fine, however you can't have the contrary; there is at least one eigenvector per eigenvalue (otherwise the definition of eigenvector/eigenvalue doesn't make sense).

(f) Matrices *i)*, *ii)*, and *iv)* are the only ones that have 2 linearly independent eigenvectors, hence they are the only ones that are diagonalisable.

2. (a) From the hint, we know that $\Lambda e^{\Lambda t} = e^{\Lambda t}\Lambda$. Thus, we only need to show $\frac{d}{dt}e^{\Lambda t} = \Lambda e^{\Lambda t}$ (or, equivalently, $\frac{d}{dt}e^{\Lambda t} = e^{\Lambda t}\Lambda$):

$$\frac{d}{dt}e^{\Lambda t} = \begin{bmatrix} \frac{d}{dt}e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & \frac{d}{dt}e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{d}{dt}e^{\lambda_n t} \end{bmatrix} = \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & \lambda_2 e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n e^{\lambda_n t} \end{bmatrix} = \Lambda e^{\Lambda t}.$$

(b) Starting from rule **I.**

$$\dot{x}(t) = Ax \Leftrightarrow \dot{x}(t) - Ax(t) = 0 \Leftrightarrow (Ve^{-\Lambda t}V^{-1})(\dot{x}(t) - Ax(t)) = 0.$$

Notice that the fact that $(Ve^{-\Lambda t}V^{-1})$ is invertible (with inverse $(Ve^{-\Lambda t}V^{-1})^{-1} = Ve^{\Lambda t}V^{-1}$) is responsible for the second if and only if, i.e., the second $\Leftrightarrow$. Next,

$$(Ve^{-\Lambda t}V^{-1})(\dot{x}(t) - Ax(t)) = Ve^{-\Lambda t}V^{-1}\dot{x}(t) - Ve^{-\Lambda t}V^{-1}Ax(t).$$

But, $A = V\Lambda V^{-1}$, so

$$-Ve^{-\Lambda t}V^{-1}Ax(t) = -Ve^{-\Lambda t}V^{-1}V\Lambda V^{-1}x(t) = -Ve^{-\Lambda t}\Lambda V^{-1}x(t) = Ve^{-\Lambda t}(-\Lambda)V^{-1}x(t).$$

From part (a) we have that $\frac{d}{dt}e^{(-\Lambda)t} = e^{\Lambda t}(-\Lambda)$. So,

$$Ve^{-\Lambda t}(-\Lambda)V^{-1}x(t) = V\frac{d}{dt}(e^{(-\Lambda)t})V^{-1}x(t).$$

Using the second hint we have that $V\frac{d}{dt}e^{(-\Lambda)t}V^{-1} = \frac{d}{dt}(Ve^{(-\Lambda)t}V^{-1})$ and thus

$$\begin{aligned} Ve^{-\Lambda t}V^{-1}\dot{x}(t) - Ve^{-\Lambda t}V^{-1}Ax(t) &= Ve^{-\Lambda t}V^{-1}\frac{d}{dt}(x(t)) + \frac{d}{dt}(Ve^{(-\Lambda)t}V^{-1})x(t) \\ &= \frac{d}{dt}(Ve^{(-\Lambda)t}V^{-1}x(t)). \end{aligned}$$

$$\dot{x}(t) = Ax \Leftrightarrow \frac{d}{dt}(Ve^{(-\Lambda)t}V^{-1}x(t)) = 0 \Leftrightarrow \int \frac{d}{dt}(Ve^{(-\Lambda)t}V^{-1}x(t))dt = Ve^{(-\Lambda)t}V^{-1}x(t) = c$$

where $c \in \mathbb{R}^n$ is a vector of integration constants. Then, pre-multiplying the above with $Ve^{(\Lambda)t}V^{-1}$ we get that $x(t) = Ve^{\Lambda t}V^{-1}c$.

(c)

$$e^{\Lambda 0} = \begin{bmatrix} e^{\lambda_1 0} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 0} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n 0} \end{bmatrix} = \begin{bmatrix} e^0 & 0 & \dots & 0 \\ 0 & e^0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I.$$

(d) $x(0) = Ve^{\Lambda 0}V^{-1}c = VIV^{-1}x_0 = VV^{-1}c = Ic$. But, $x(0) = x_0$, hence, $c = x_0$.

(e) In parts $(a) - (d)$ we proved that $x(t) = Ve^{\Lambda t}V^{-1}x_0$ is a solution to (1). In addition, given that throughout the whole derivation we only employed if and only ifs, $\Leftrightarrow$, we also proved ("for free!") that $x(t) = Ve^{\Lambda t}V^{-1}x_0$ is the only solution to (1).

3. (a) By assumption, we have that $\dot{x}(t) = Ax(t)$, $x(0) = x_0$, $\dot{z}(t) = Az(t)$, $z(0) = x_0$ and $x \not\equiv z$, i.e., that there exists a $t \geq 0$ such that $x(t) \neq z(t)$. Note that

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)d\tau = x_0 + \int_0^t Ax(\tau)d\tau, \qquad z(t) = z(0) + \int_0^t \dot{z}(\tau)d\tau = x_0 + \int_0^t Az(\tau)d\tau.$$

Hence, using both linearity of integration and the properties of norms,

$$\begin{aligned} \delta(t) = ||x(t) - z(t)|| &= \left|\left|x_0 + \int_0^t Ax(\tau)d\tau - \left(x_0 + \int_0^t Az(\tau)d\tau\right)\right|\right| = \left|\left|\int_0^t A(x(\tau) - z(\tau))d\tau\right|\right| \\ &\leq \int_0^t ||A(x(\tau) - z(\tau))||d\tau \leq \int_0^t ||A||\ ||x(\tau) - z(\tau)||d\tau = \int_0^t ||A||\delta(t)\tau. \end{aligned}$$

Using $\alpha = 0$ and $\beta = ||A|| \geq 0$ and applying (3) we get

$$\delta(t) \leq \alpha e^{\beta t} = 0e^{||A||t} = 0, \quad \forall t \geq 0.$$

Hence, $\delta \equiv 0$, so $x \equiv z$, which contradicts our initial assumption that $x(t)$ and $z(t)$ are different (at least for one value of t). Thus, there are not multiple solutions to (1), there is at most one.

(b)

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{d}{dt}(Ve^{\Lambda t}V^{-1}x_0) = V\frac{d}{dt}(e^{\Lambda t})V^{-1}x_0 = V\Lambda e^{\Lambda t}V^{-1}x_0$$

But $A = V\Lambda V^{-1}$, hence $AV = V\Lambda$. Plugging this into the above we get $\dot{x}(t) = AVe^{\Lambda t}V^{-1}x_0 = Ax(t)$.

(c) $x(0) = Ve^{\Lambda 0}V^{-1}x_0 = VIV^{-1}x_0 = VV^{-1}x_0 = Ix_0 = x_0$. Thus we can conclude that the unique solution to (1) is given by (2).