Problem Sheet 3: Second order linear models & Romeo and Juliet

If you find any typos/errors in this problem sheet please email jk208@ic.ac.uk.

1. In the lecture notes and in Problem Sheet 2 we argued that the solution to the 2 dimensional model
$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{R}^2, \tag{1}$$
is given by
$$x(t) = Ve^{\Lambda t}V^{-1}x_0 \tag{2}$$
where V is the matrix whose columns are the eigenvectors of A, Λ is the diagonal matrix with $\Lambda_{ii} = \lambda_i$ being the eigenvalue corresponding to the i^{th} eigenvector and
$$e^{\Lambda t} := \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$
In this exercise we study how different matrices A lead to different behaviours of $x(t)$. This (light-hearted) exercise was proposed by Strogatz [Strogatz, 1988, 1994]. He gives a (rather geeky) twist to the tale of Romeo and Juliet...

"Romeo is in love with Juliet, but in our version of this story Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him. Let
$$R(t) = \text{Romeo's love/hate for Juliet at time } t,$$
$$J(t) = \text{Juliet's love/hate for Romeo at time } t.$$
Positive values of R, J signify love, negative values signify hate. Then a model for their star-crossed romance is"
$$\begin{aligned} \dot{R}(t) &= J(t) \qquad & R(0) &= R_0 \\ \dot{J}(t) &= -R(t) \qquad & J(0) &= J_0 \end{aligned} \tag{3}$$

   (a) Re-write (3) in the form $\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = A \begin{bmatrix} R \\ J \end{bmatrix}$ for some matrix A.

   (b) Work out the eigenvectors and eigenvalues of A.

   (c) Find V^{-1}.

   (d) Use (2) to express $R(t)$ and $J(t)$ in terms of e^{it}, e^{-it}, R_0 and J_0.

   (e) Use Euler's formula to show that
$$R(t) = R_0 \cos(t) + J_0 \sin(t)$$
$$J(t) = J_0 \cos(t) - R_0 \sin(t).$$

   (f) Work out the quantity $R(t)^2 + J(t)^2$. Does it depend on t? What does this mean with regards to the shape of the trajectory of the phase plane, i.e., the graph of $J(t)$ vs $R(t)$?

   (g) Use the previous part to sketch $J(t)$ vs $R(t)$ for different values of $R_0^2 + J_0^2$ (this is called a *phase portrait*). *Hint: by evaluating* (3) *at carefully chosen values of R and J you can deduce in which direction the trajectories are going.*

   (h) Interpret the phase portrait in terms of Romeo and Juliet's relationship.

2. Let's now examine some other Romeo and Juliet pairs and try to predict the outcome of the relationship.

$$i)\quad \begin{aligned}\dot{R} &= 3R + J \\ \dot{J} &= R + 3J\end{aligned} \qquad ii)\quad \begin{aligned}\dot{R} &= -J \\ \dot{J} &= -R\end{aligned} \qquad iii)\quad \begin{aligned}\dot{R} &= -2R + J \\ \dot{J} &= R - 2J\end{aligned}$$

In *i)* both Romeo and Juliet echo each others feelings. If one loves/hates the other, the other's love/hate for the first grows. However, both Romeo and Juliet are quite self involved and they care 3 times as much about their own feelings regarding the other than about the other's feelings regarding them. In other words they are both loyal and they both hold a grudge; if Romeo (Juliet) has strong positive/negative feelings for Juliet (Romeo), then unless Juliet (Romeo) exhibits an extraordinary amount of disdain/affection towards them, sufficient to change their mind, these feelings will persist and grow.

In *ii)* both Romeo and Juliet are, as Strogatz describes it, "fickle lovers".

In *iii)* both Romeo and Juliet are very cautious people. They do echo each others feeling, however they are quite wary of growing strong feelings towards each other.

For all three cases:

(a) Re-write the systems as $\begin{bmatrix}\dot{R}\\ \dot{J}\end{bmatrix} = A\begin{bmatrix}R\\ J\end{bmatrix}$.

(b) Work out the eigenvectors and eigenvalues.

(c) Using

$$\begin{bmatrix}R(t)\\ J(t)\end{bmatrix} = Ve^{\Lambda t}V^{-1}\begin{bmatrix}R_0\\ J_0\end{bmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

where c_1 and c_2 are some constants[1] such that $R(0) = R_0$ and $J(0) = J_0$, draw the phase portraits. *Hint 1: you do not need to work out c_1 and c_2 explicitly. Hint 2: Two (or more!) trajectories can only cross at a fixed point.*

(d) Interpret what happens in terms of Romeo and Juliet's relationship.

3. Now consider these final two Romeo and Juliet pairs

$$i)\quad \begin{aligned}\dot{R} &= J - R \\ \dot{J} &= -R - J\end{aligned} \qquad ii)\quad \begin{aligned}\dot{R} &= R + J \\ \dot{J} &= J - R\end{aligned}$$

These are really just modified version of the original Romeo and Juliet pair we looked at in exercise 2. In *i)* the difference is that both Romeo and Juliet are rather cautious and are wary of letting their feelings for each other grow too much. In *ii)* the difference is that both Romeo and Juliet find it hard to let go of their initial feelings for one another. Indeed, if they have no contact with each other, they remain fixated on their initial impressions, and thus their initial feelings, whether positive or negative, grow.

To study these we could proceed as normal, using eigenvalues/eigenvectors, Euler's formula, obtaining the analytical solution and deducing what will happen directly from the solution (try doing it as an exercise!). However, we can try a different approach in this exercise. We know that these two Romeo and Juliet pairs are just modified versions of those we saw in exercise 2. In addition, we

[1] Remember these come from $V^{-1}\begin{bmatrix}R_0\\ J_0\end{bmatrix}$.

now that the original Romeo and Juliet model ended up oscillating, hence translating the model into polar coordinates seems like a reasonable thing to do[2].

(a) If we let $r(t)$ denote the distance from the origin and $\theta(t)$ the angle between the vector $\begin{bmatrix} R(t) \\ J(t) \end{bmatrix}$ and the horizontal axis we have that

$$r(t)^2 = R(t)^2 + J(t)^2, \quad \text{and} \quad \theta(t) = \arctan\left(\frac{J(t)}{R(t)}\right).$$

Use the above to show that in Case *i)* $\dot{r}(t) = -r(t)$, $\dot{\theta}(t) = -1$ and in Case *ii)* $\dot{r}(t) = r(t)$, $\dot{\theta}(t) = -1$. *Hint: take derivatives with respect to time of both sides of the equations, use the chain rule and remember that* $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

(b) For both cases, use part (a) to find expressions for $r(t)$ and $\theta(t)$ in terms of t, R_0 and J_0. Use these to sketch the phase portraits. *Hint: use an integrating factor to work out* $r(t)$.

4. Consider a general n dimensional model defined by

$$\dot{x} = Ax \tag{4}$$

where A is an $n \times n$ matrix. Suppose that A is diagonalisable.

(a) Find all fixed points of (4). *Hint: If A is diagonalisable what is the rank of A? Does its inverse exists? What does this imply regarding the number of vectors x we can find such that $Ax = y$ if y is given? Can you find a single vector x such that $Ax = 0$ for any A?*

(b) Remind yourselves of the definitions of stable/unstable nodes, stable/unstable spiral, saddle point and centre given in the lecture notes (section 5.4). Use the phase portraits you have sketched throughout exercises $1-3$ to classify the origin of each of the six Romeo and Juliet pairs.

References

Steven H Strogatz. Love affairs and differential equations. *Mathematics Magazine*, 61:35, 1988.

Steven H. Strogatz. *Nonlinear Dynamics and Chaos.* Addison-Wesley, Reading, MA, 1994.

[2]For general $2-D$ models, as soon as you see complex eigenvalues you can start thinking about doing this. However, often it will be considerably more messy than in this exercise. One can avoid the messiness by first doing the change of coordinates $z = V^{-1}x$ and then working out the z system in polar coordinates (**however, you will not be asked to do this in any course assessment**).

Solutions

1. (a) $\begin{bmatrix}\dot{R}\\ \dot{J}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}R\\ J\end{bmatrix}$.

   (b) The eigenvalues are given by the solution to $\det(A - I\lambda) = 0$, that is, $\lambda^2 + 1 = 0$. Hence, $\lambda_1 = i$ and $\lambda_2 = -i$ (or the other way round, all that matters is that later on you are consistent in where you place the eigenvectors in V). Then, eigenvectors are given by $v_1 = [i, -1]^T$, $v_2 = [i, 1]^T$ (or any scaled version of these, e.g., $v_1 = [3i, -3]^T$ and $v_2 = [4i, 4]^T$).

   (c) Using $V = [v_1, v_2]$ we have that $V^{-1} = \begin{bmatrix}-\frac{i}{2} & -\frac{1}{2}\\ -\frac{i}{2} & \frac{1}{2}\end{bmatrix}$.

   (d)
   $$\begin{bmatrix}R(t)\\ J(t)\end{bmatrix} = Ve^{\Lambda t}V^{-1}x_0 = \begin{bmatrix}i & i\\ -1 & 1\end{bmatrix}\begin{bmatrix}e^{it} & 0\\ 0 & e^{-it}\end{bmatrix}\begin{bmatrix}-\frac{i}{2} & -\frac{1}{2}\\ -\frac{i}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}R_0\\ J_0\end{bmatrix}.$$
   $$\Rightarrow \begin{aligned} R(t) &= \left(\tfrac{1}{2}R_0 - \tfrac{i}{2}J_0\right)e^{it} + \left(\tfrac{1}{2}R_0 + \tfrac{i}{2}J_0\right)e^{-it}\\ J(t) &= \left(\tfrac{i}{2}R_0 + \tfrac{1}{2}J_0\right)e^{it} + \left(-\tfrac{i}{2}R_0 + \tfrac{1}{2}J_0\right)e^{-it}\end{aligned}$$

   (e) Euler's formula is $e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$. So
   $$R(t) = \left(\frac{1}{2}R_0 - \frac{i}{2}J_0\right)(\cos(t) + i\sin(t)) + \left(\frac{1}{2}R_0 + \frac{i}{2}J_0\right)(\cos(t) - i\sin(t)) = R_0\cos(t) + J_0\sin(t).$$
   and
   $$J(t) = \left(\frac{i}{2}R_0 + \frac{1}{2}J_0\right)(\cos(t) + i\sin(t)) + \left(-\frac{i}{2}R_0 + \frac{1}{2}J_0\right)(\cos(t) - i\sin(t)) = J_0\cos(t) - R_0\sin(t).$$

   (f) Remember that $sin^2(t) + cos^2(t) = 1$, so
   $$R^2(t) + J^2(t) = (R_0\cos(t) + J_0\sin(t))^2 + (J_0\cos(t) - R_0\sin(t))^2 = (R_0^2 + J_0^2)(sin^2(t) + cos^2(t)) = R_0^2 + J_0^2.$$
   By Pythagoras' Theorem, we have that the distance from the origin is given by $r(t) = \sqrt{R^2(t) + J^2(t)} = \sqrt{R_0^2 + J_0^2}$. Hence, the trajectories are closed orbits (more specifically circles) in the J vs R plane!

   (g) Given the above, all that is left to deduce is whether the trajectories are rotating in the clockwise or anticlockwise direction. Examining, $R(t)$ and $J(t)$ will give you the answer. However, in general, we don't have analytic solutions for arbitrary models. However, if we can show that the trajectories are confined to a closed orbit (as we will do later in the course), we can use the model definition (3) to find out in which direction they are travelling. Consider the point $(a, 0)$ with $a > 0$. Then the velocity vector is $\begin{bmatrix}\dot{R}\\ \dot{J}\end{bmatrix} = \begin{bmatrix}0\\ -a\end{bmatrix}$, which is pointing straight down, hence the trajectories are rotating in the clockwise direction.

   So, the phase portrait should look like the one in Figure 1.

   (h) As Strogatz describes, the story of Romeo and Juliet is still a tragedy; "The sad outcome of their affair is, of course, a never-ending cycle of love and hate; the governing equations are those of a simple harmonic oscillator. At least they manage to achieve simultaneous love one-quarter of the time."

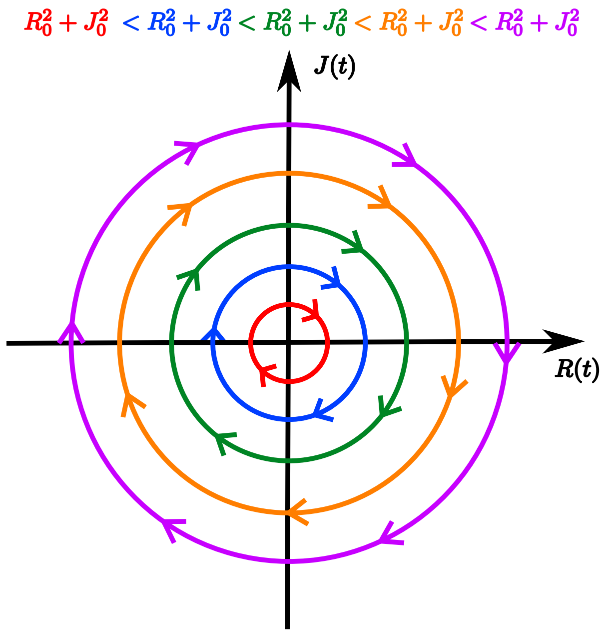

Figure 1: Phase portrait.

2. **Case** i)

   (a) Re-writing, we get

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

   (b) The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 4$ and the eigenvectors are $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (c) We have that

$$\begin{bmatrix} R(t) \\ J(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}.$$

   Notice that as t increases, e^{4t} grows much quicker than e^{2t}. So unless $c_2 = 0$, after a while we will have $|c_2 e^{4t}| >> |c_1 e^{2t}|$, so the trajectories will be parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If $c_2 = 0$, then $\begin{bmatrix} R(0) \\ J(0) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. In other words, if $c_2 = 0$, then the initial conditions must lie on $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. In that case they will simply keep growing along $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So the phase portrait should look like that in Figure 2.

   (d) In this case, Romeo and Juliet echo each others feelings. If one loves (hates) the other, the other's love (hate) for the first increases. Thus, if initially they both love (hate) each other, i.e., the initial conditions are in the 1^{st} (3^{rd}) quadrant, then their mutual love (hate) increases forever. If this not the case; if initially one likes the other but the other dislikes the first, then the outcome of the story will depend on whether the first's positive feelings for the second are stronger than the second's negative feelings for the first. In our phase plane we can see this by observing that if the trajectory starts "above" the line $R = -J$, in other words "above" v_1 or $R_0 + J_0 > 0$, then it will eventually keep in growing in the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ direction. If this is not the case, Romeo and Juliet seem to be heading straight towards a very large fight...

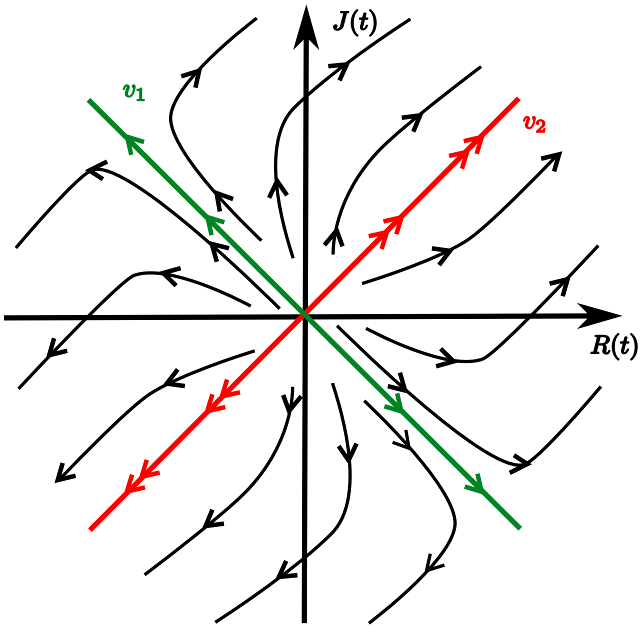

Figure 2: Phase portrait for case i).

Case ii)

(a) Re-writing we get

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

(b) The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$ and the eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(c) We have that

$$\begin{bmatrix} R(t) \\ J(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t}.$$

As time passes we have that $c_1 e^{-t} \to 0$ and $c_2 e^t \to \pm\infty$ (depending on whether c_2 is positive or negative). Hence, as $t \to +\infty$, the component of the trajectory in the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ dies out, while that in the $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ keeps growing forever. So, unless $c_2 = 0$, the trajectory will tend towards $\pm\infty$ in the v_2 direction. If $c_2 = 0$, then the initial conditions must lie on $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In that case the trajectory will simply converge to zero along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So the phase portrait should look like that in Figure 3.

(d) This relationship seems to be doomed to fail from the start. They each want what they can't have and get bored of what they can have. So if one dislikes the other, the affection of the second for the first grows and vice versa. For this reason regardless of what the initial feelings were, one will eventually end up increasingly rejecting the other while the other ("strangely attracted" by the rejection) will ever more desperately throw themselves at the first (which will further lower the first's opinion of the second). Eventually, one will have to get a restraining order on the other...

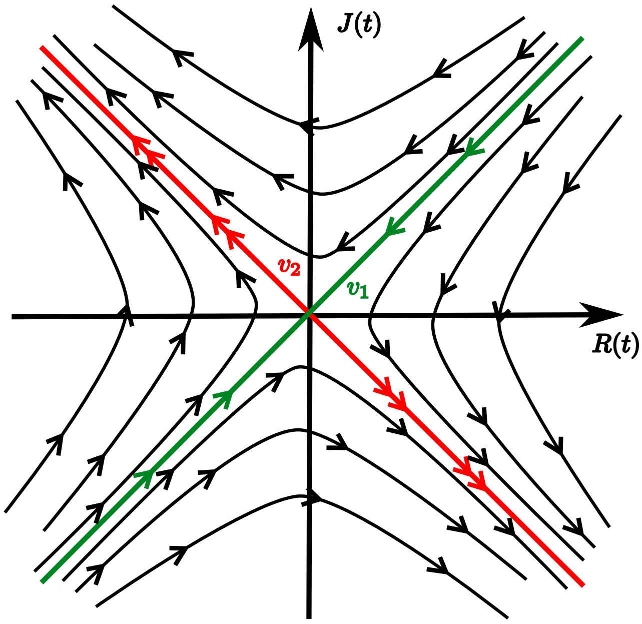

Figure 3: Phase portrait for case *ii*).

Case *iii*)

(a) Re-writing we get

$$\begin{bmatrix}\dot{R}\\ \dot{J}\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}R\\ J\end{bmatrix}.$$

(b) The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -3$ and the eigenvectors are $v_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

(c) We have that

$$\begin{bmatrix}R(t)\\ J(t)\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-t} + c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{-3t}.$$

As time passes we have that $c_1e^{-t} \to 0$ and $c_2e^{-3t} \to 0$. Hence, regardless of the initial conditions we have that $\begin{bmatrix}R(t)\\ J(t)\end{bmatrix} \to \begin{bmatrix}0\\0\end{bmatrix}$ as $t \to +\infty$. However, e^{-3t} tends to zero much faster than e^{-t}, hence the $\begin{bmatrix}-1\\1\end{bmatrix}$ component of the trajectory dies out much quicker than the $\begin{bmatrix}1\\1\end{bmatrix}$ does. So the phase portrait should look like that in Figure 4.

(d) In this case both Romeo and Juliet are way too cautious and even in the best scenario in which they both fancy each other, they are too afraid to act on their feelings. Hence any initial feelings they have eventually die out and all that remains is mutual indifference. As Strogatz says "the lesson seems to be that excessive caution can lead to apathy".

3. (a) For Case *i*) we have that

$$\frac{d}{dt}r^2 = \frac{d}{dt}(R^2+J^2) \Rightarrow 2\dot{r}r = 2\dot{R}R+2\dot{J}J \Rightarrow \dot{r} = \frac{(-R+J)R+(-R-J)J}{r} = \frac{-R^2-J^2}{r} = -\frac{r^2}{r} = -r.$$

and

$$\dot{\theta} = \frac{d}{dt}\theta = \frac{d}{dt}\arctan\left(\frac{J}{R}\right) = \frac{1}{1+\left(\frac{J}{R}\right)^2}\frac{d}{dt}\left(\frac{J}{R}\right) = \frac{1}{1+\left(\frac{J}{R}\right)^2}\frac{\dot{J}R-J\dot{R}}{R^2}$$

$$= \frac{R^2}{R^2+J^2}\frac{(-R-J)R-J(-R+J)}{R^2} = -1.$$

The derivation is similar for Case *ii*) and yields $\dot{r} = r$ and $\dot{\theta} = -1$

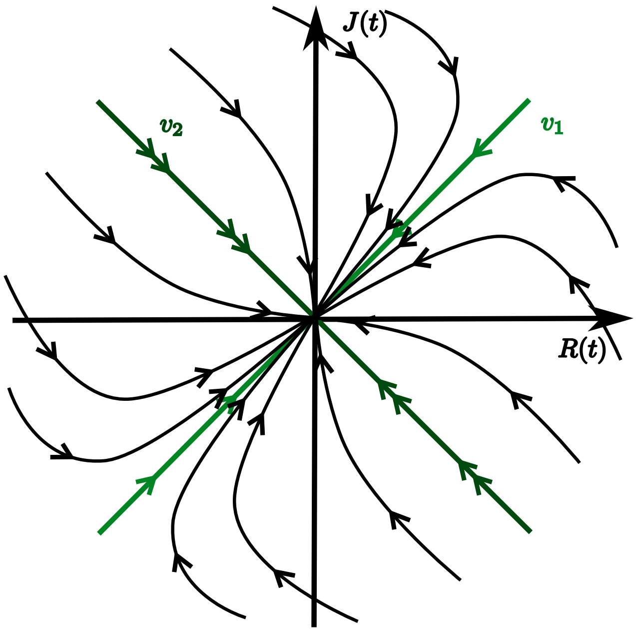

Figure 4: Phase portrait for case *iii*).

(b) For Case *i*) we have that

$$\dot{r} = -r \Rightarrow \frac{dr}{dt} = -r \Rightarrow \frac{dr}{dt} + r = 0 \Rightarrow \frac{dr}{dt}e^t + re^t = \frac{d}{dt}\left(re^t\right) = 0 \Rightarrow r(t)e^t = C \Rightarrow r(t) = Ce^{-t},$$

where C is some integration constant. We know that $r(0) = C = \sqrt{R(0)^2 + J(0)^2} = \sqrt{R_0^2 + J_0^2}$.
$\theta(t)$ is straightforward to obtain. We have

$$\dot{\theta}(t) = -1 \Rightarrow \theta(t) = \int \dot{\theta}(t)dt = \int -1 = C - t$$

Using the initial conditions, $\theta(0) = C = \arctan\left(\frac{J_0}{R_0}\right)$.

Similarly, for Case *ii*) we get that $r(t) = (\sqrt{R_0^2 + J_0^2})e^t$ and $\theta(t) = \arctan\left(\frac{J_0}{R_0}\right) - t$. With regards to the phase portraits $r(t)$ and $\theta(t)$ tell us all we need. In Case *i*), the distance from the origin, $r(t)$ is decreasing exponentially with time, while in Case *ii*) it is increasing exponentially with time. For both cases the angle between the $\begin{bmatrix} R(t) \\ J(t) \end{bmatrix}$ vector and the horizontal axis is decreasing linearly with time. Hence in Case *i*) we get the trajectories spiralling inwards (rotating clockwise) towards the origin, see Figure 5a, while in Case *ii*) they are spiralling outwards towards infinity, see Figure 5b.

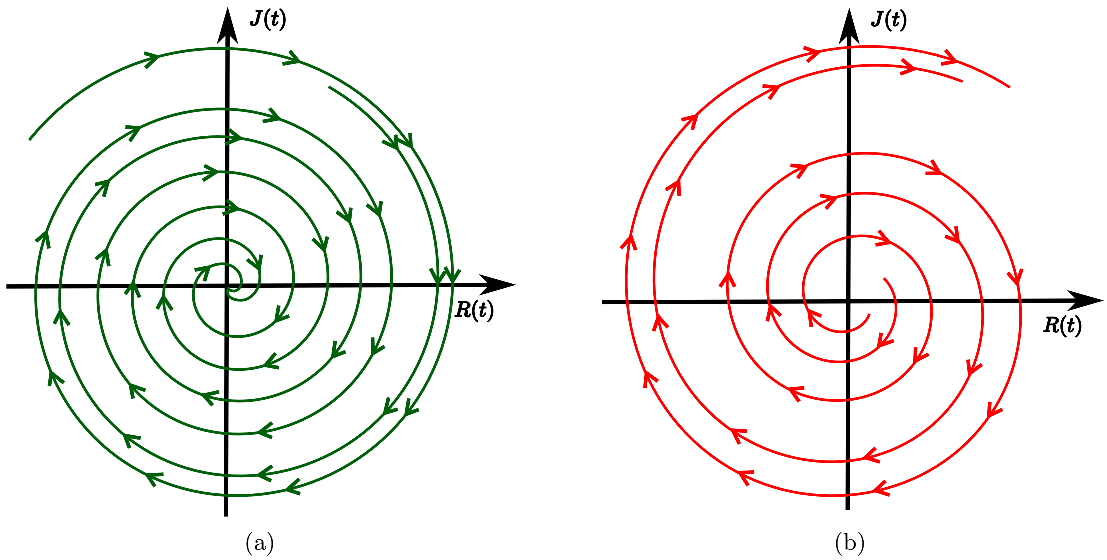

Figure 5: (a) Phase portrait for Case *i*). (b) Phase portrait for Case *ii*).

4. (a) A fixed point of a model $\dot{x} = f(x)$ is a point $\bar{x}$ such that $f(\bar{x}) = 0$. So we are interested in vectors $\bar{x}$ such that $A\bar{x} = 0$. By definition A is diagonalisable if and only if it has n linearly independent eigenvectors. As discussed in exercise 1, part (f) of Problem Sheet 2 the number of linearly independent eigenvectors of A is less or equal to the rank of A. Since A has rank n, we know that A has an inverse A^{-1} (a matrix which has full rank is such that its determinant is nonzero and is thus always invertible; this is also related to what is discussed in exercise 1 part (b) of Problem Sheet 2). So for a given vector y there is a unique vector x such that $Ax = y$. This vector is $x = A^{-1}y$. Thus there is only a single vector $\bar{x}$ such that $A\bar{x} = 0$, that is, there is a single fixed point. Noticing that $A0 = 0$ (0 in this context denotes the vector of zeros), we conclude that $\bar{x} = 0$ is the only fixed point.

   (b) Exercise 1: Centre. Exercise 2: *i*) Unstable node, *ii*) saddle point, *iii*) stable node. Exercise 3: *i*) Stable spiral, *ii*) unstable spiral.