Imperial College London

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Warm-up exercises (to be done as homework)

The exercises that follow are meant to help you refresh your applied knowledge of simple mathematical concepts used throughout the course.

Exercise 1: Differentiating and drawing functions

Consider the following functions:

- $f(x) = -x^2$
- $g(x) = \frac{1}{x}$
- $h(x) = \frac{1}{1+x^2}$
- $k(x) = \frac{x^2}{1+x^2}$

1. Differentiate these functions with respect to x analytically, i.e., write down the analytical expressions of $f'(x)$, $g'(x)$, $h'(x)$, and $k'(x)$.
2. Draw these functions by considering the asymptotic behaviours for $x \to \infty$, $x \to -\infty$, the zero-crossing points (values of x at which these functions are zero), the values at which the derivatives are zero (which correspond to extrema of the function), and the values at which the second derivatives are zero (which correspond to points of inflexion).

Exercise 2: Calculating the roots of quadratic equations

Calculate by hand the (complex) roots of these quadratic equations:

- $w^2 + 1 = 0$
- $x^2 + x + 1 = 0$
- $5y^2 + 2y + 3 = 0$
- $-2z^2 + 3z - 1 = 0$

Exercise 3: Scalars, vectors, matrices and eigenvalues

Consider the vectors:

$$a = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$$

1. Calculate by hand the following expressions:

- $a^T b$
- ab^T
- $ab^T ba^T$
- $ab^T a$
- $\left(ba^T\right)^T$
- $b^T ab^T$
- calculate the angle in degrees between a and b
- write down the expression of a vector orthogonal to the plane spanned by a and b.
- based on the vector you have calculated in the previous point, write down the equation of the plane which has this vector as its normal and which is at a distance of 1 of the origin.

where ^{T} denotes the transposition operator.

2. Calculate by hand the eigenvalues and normalised eigenvectors of the following matrices:

- $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
- $B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$
- $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$

Exercise 4: Solving a system of linear equations

Solve the following linear system for x:

$$Ax = y$$

when

- $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
- $A = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$

(Hint, the question mainly concerns how you compute A^{-1}?)

Exercise 5: Basic combinatorics

Show your calculation to the following two questions:

- Consider an urn with 6 balls numbered $1 \ldots 6$. How many 2-element sets of balls can be drawn out of the urn if balls are not placed back in the urn after each draw?
- Consider an urn with 6 balls numbered $1 \ldots 6$. How many 2-element sets of balls can be drawn out of the urn if balls can be placed back in the urn immediately after each draw?

Solutions

Question 1 • $f'(x) = -2x$, $g'(x) = -\frac{1}{x^2}$, $h'(x) = -\frac{2x}{(1+x^2)^2}$, $k'(x) = \frac{2x}{(1+x^2)^2}$

Question 2 • $w = \pm i$, $x = \frac{-1\pm i\sqrt{3}}{2}$, $y = \frac{-2\pm i\sqrt{56}}{10}$, $z = \{1, 0.5\}$

Question 3.1 • $a^T b = 9$, $ab^T = \begin{pmatrix} 2 & -5 & -1 \\ -4 & 10 & 2 \\ 6 & -15 & -3 \end{pmatrix}$, $ab^T ba^T = \begin{pmatrix} 30 & -60 & 90 \\ -60 & 120 & -180 \\ 90 & -180 & 270 \end{pmatrix}$, $ab^T a = \begin{pmatrix} 9 \\ -18 \\ 27 \end{pmatrix}$, $(ba^T)^T = 9$, $b^T ab^T = \begin{pmatrix} 18 & -45 & -9 \end{pmatrix}$, $\angle(a,b) = \arccos\left(\frac{a^T b}{\|a\|\|b\|}\right) = 1.1161\ rad = 63.95$ deg.

A vector orthogonal to the plane spanned by a and b is a scalar multiple of $a \times b = \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix}$.

The equation for the plane is $17x + 7y - z + D = 0$. After normalising the normal vector we get $0.92x + 0.38y - 0.054z + d = 0$, and therefore the plane with distance 1 from the origin is $0.92x + 0.38y - 0.054z + 1 = 0$.

Question 3.2

matrix	eigenvalues	eigenvectors
$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\lambda_1 = 1$, $\lambda_2 = 3$	$v_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $v_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
$B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$	$\lambda_1 = 2$, $\lambda_2 = 2$	$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$	$\lambda_1 = 2i$, $\lambda_2 = -2i$, $\lambda_3 = 1$	$v_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ -1 \\ -i \end{pmatrix}$, $v_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ -1 \\ i \end{pmatrix}$, $v_3 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Question 4 Provided the inverse A^{-1} of the matrix A exists, the solution is given by $x = A^{-1}y$. For a matrix of the form $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is given by $D^{-1} = \frac{1}{\det(D)}\begin{pmatrix} -a & c \\ b & -d \end{pmatrix}$ with $\det(D) = ad - bc$.

Question 5 • $\frac{n!}{k!(n-k)!}$ with $n = 6$ and $k = 2$ gives a total of 15 combinations.

• Adding the 6 pairs of repeated balls, the total is 21.