Part IIB/EIST Part II, Module 4F2

Robust Multivariable Control

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HANDOUT 2

Infinite horizons and the $\mathcal{H}_2$ norm

References

1. Green and Limebeer, "Linear Robust Control", Prentice Hall, 1995

2. Anderson and Moore, "Optimal Control: Linear Quadratic Methods", Prentice Hall, 1990

Contents

2.1 Infinite Horizon Linear Quadratic Regulator

Plant:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$z = \begin{bmatrix} Cx \\ u \end{bmatrix}$$

Cost Function:

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty z(t)^T z(t)\, dt \\ &= \int_0^\infty \left(x(t)^T C^T C x(t) + u(t)^T u(t) \right) dt \end{aligned}$$

Assumptions:
A, B controllable
A, C observable
Solution:
From the finite horizon results, and our understanding of the Riccati equation, we would expect the solution to be of the form

$$u(t) = -B^T X x(t)$$

where $X = X^T$ solves the Control Algebraic Riccati Equation

$$0 = C^T C + XA + A^T X - XBB^T X \qquad \text{(CARE)}$$

The closed-loop dynamics would then be governed by

$$\dot{x} = Ax + Bu = (A - BB^T X)x$$

– we might hope that $(A - BB^T X)$ is stable (i.e. has all its eigenvalues in the left half plane.)

Fact: Under the assumptions, the CARE

$$0 = C^T C + XA + A^T X - XBB^T X$$

has a unique, symmetric, positive definite solution $X = X^T > 0$, and this solution is *stabilising* (i.e. $(A - BB^T X)$ stable).
Furthermore, this solution can be obtained as $\lim_{t \to -\infty} X(t)$, where $X(t)$ solves

$$-\dot{X}(t) = C^T C + X(t)A + A^T X(t) - X(t)BB^T X(t)$$

for *any* final condition $X(T) = X^T(T) > 0$.

Summary: Let $X = X^T$ be the stabilising solution to CARE. Then the optimal control is given by $u(t) = -B^T X x(t)$ and the optimal cost is $x(0)^T X x(0)$.

Alternative Derivation: *(more direct, but you have to already know the answer!)*
Let $X = X^T$ be the stabilising solution to CARE, and consider

$$V(t) = x^T(t) X x(t) \implies \frac{dV}{dt} = \dot{x}^T X x + x^T X \dot{x}$$

So,

$$\begin{aligned}\frac{dV}{dt} + z^T z \quad &= \\ &= (Ax + Bu)^T X x + x^T X (Ax + Bu) + x^T C^T C x + u^T u \\ &= (u + B^T X x)^T (u + B^T X x) + x^T \underbrace{(XA + A^T X + C^T C - XBB^T X)}\, x\end{aligned}$$

Integrating both sides of this expression, from $t = 0$ to ∞, gives

$$V(\infty) - \underbrace{V(0)}_{x_0^T X x_0} + \|z\|_2^2 = \|(u + B^T X x)\|_2^2$$

Or

$$\|z\|_2^2 = x(0)^T X x(0) + \|(u + B^T X x)\|_2^2$$

Note: If all the states are not available for measurement (i.e. we are not in the state feedback situation), then we see that we have to make $\|(u + B^T X x)\|_2$ small. To do this we use a Kalman filter to estimate $-B^T X x$ – this leads to LQG (linear quadratic Gaussian) control, which is a special case of $\mathcal{H}_2$ optimal control.

2.2 The $\mathcal{H}_2$-norm

Consider the stable linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where A has all its eigenvalues in the left half plane. This system has a transfer function

$$\hat{G}(s) = C(sI - A)^{-1}B$$

The $\mathcal{H}_2$ norm of this system is defined as

$$\|\hat{G}\|_2^2 = \int_{-\infty}^{\infty} \text{trace}\left\{\hat{G}^*(j\omega)\hat{G}(j\omega)\right\} d\omega$$

and so

$$\|\hat{G}\|_2^2 = \sum_i \|\hat{G}_i\|_2^2$$

One can show that

$$\|y\|_\infty \leq \frac{1}{\sqrt{2\pi}}\|\hat{G}\|_2\|u\|_2$$

where

$$\|y\|_\infty = \sup_t \sqrt{y^T(t)y(t)}$$

and

$$\|u\|_2 = \sqrt{\int_{-\infty}^{\infty} u^T(t)u(t)dt}$$

The aim of $\mathcal{H}_2$ optimal control is to minimise the $\mathcal{H}_2$ norm of some closed-loop transfer function matrix.

2.3 Calculating the $\mathcal{H}_2$-norm

Let the impulse response matrix of $\hat{G}(s)$ be $G(t)$. Recall that

$$G(t) = \mathcal{L}^{-1}\hat{G}(s) \\ = Ce^{At}B$$

Parseval's Theorem implies that

$$\frac{1}{\sqrt{2\pi}}\|\hat{G}(s)\|_2 = \|G(t)\|_2,$$

where

$$\|G(t)\|_2^2 = \sum_i \|G_i(t)\|_2^2$$

Since $G(t)$ is the impulse response matrix, $G_i(t)$ is the response to an impulse on the ith input with $x(0^-) = 0$.

Therefore, $G_i(t) = 0$ for $t < 0$, whereas for $t \geq 0$ it is equal to the response of the system starting at

$$x(0^+) = B_i$$

under input $u = 0$.

So the problem reduces to computing the response of the system under $u = 0$ starting at appropriate initial conditions $x(0) = x_0$.

Consider the function $V(t) = x(t)^T L x(t)$, $L = L^T$. Note that, if $u = 0$,

$$\dot{V}(t) + y(t)^T y(t) =$$

$$= (Ax(t))^T L x(t) + x(t)^T L A x(t) + x(t)^T C^T C x(t)$$
$$= x(t)^T (A^T L + LA + C^T C) x(t)$$

Choose $L = L^T$ such that

$$A^T L + LA + C^T C = 0$$

(Aside: It can be shown that $L \geq 0$ iff the system is stable. Moreover, if $L > 0$ then the system is also observable.)

$$\dot{V}(t) + y(t)^T y(t) = 0$$

Integrating from $t = 0$ to $t = \infty$ gives

$$[V(t)]_0^\infty + \|y\|_2^2 = 0$$

Since A is stable and $u = 0$,

$$\lim_{t \to \infty} x(t) = 0$$

Therefore

$$\lim_{t \to \infty} V(t) = \lim_{t \to \infty} x^T(t) L x(t) = 0$$

Moreover, $V(0) = x_0^T L x_0$, and so

$$\|y\|_2^2 = x_0^T L x_0$$

Hence, the response to initial conditions is bounded, and

$$\|y_i\|_2^2 = B_i^T L B_i \implies \sum_i \|y_i(t)\|_2^2 = \sum_i (B_i^T L B_i) = \text{trace}(B^T L B)$$

Summary:

1. $\frac{1}{\sqrt{2\pi}}\left\|\hat{G}(s)\right\|_2 = \sqrt{\text{trace}(B^T L B)}$ where $L = L^T$ solves

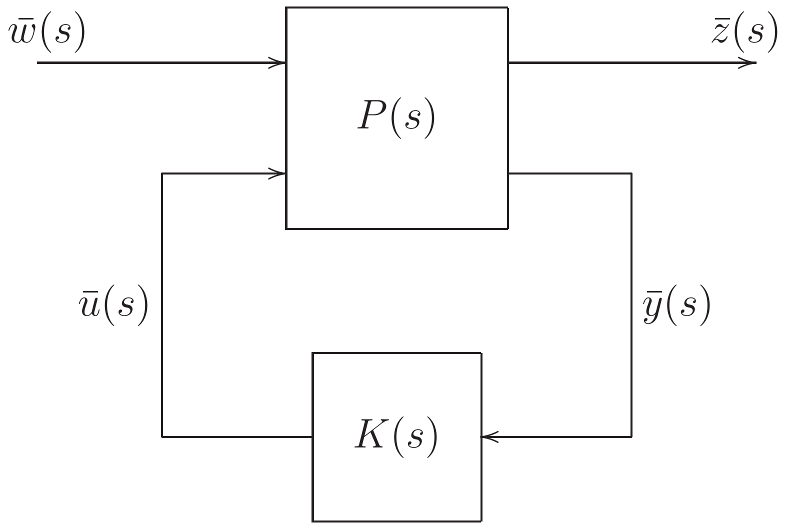
$$A^T L + LA + C^T C = 0 \quad (L: \text{ observability grammian}).$$

2. $\frac{1}{2\pi}\left\|T_{u\to y}\right\|_2^2 = \sum_i \|y(t)|_{u(t)=e_i\delta(t)}\|_2^2$

3. It can be shown that

$$\|y\|_\infty = \sup_t \sqrt{y(t)^T y(t)} \leq \frac{1}{\sqrt{2\pi}}\|T_{u\to y}\|_2\|u\|_2$$

2.4 Linear Fractional Transformations



Linear Fractional Transformations (LFT's) are a useful way of manipulating closed-loop transfer functions, and of specifying norm-optimal control problems. The lower LFT $\mathcal{F}_l(P(s), K(s))$ is defined as the closed loop transfer function from $\bar{w}(s)$ to $\bar{z}(s)$ in the above picture. That is

$$\mathcal{F}_l(P(s), K(s)) = T_{\bar{w}(s) \to \bar{z}(s)}$$

$P(s)$ is called the *Generalised Plant*.
If $P(s)$ has the *block transfer function representation*

$$\begin{bmatrix} \bar{z}(s) \\ \bar{y}(s) \end{bmatrix} = \underbrace{\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21(s)} & P_{22}(s) \end{bmatrix}}_{P(s)} \begin{bmatrix} \bar{w}(s) \\ \bar{u}(s) \end{bmatrix}$$

(where the transfer functions $P_{ij}(s)$ may themselves be matrix-valued – corresponding to vector-valued signals $\bar{w}(s)$ etc.) we then obtain

$$\begin{aligned} \bar{z}(s) &= P_{11}(s)\bar{w}(s) + P_{12}(s)\bar{u}(s) \\ \bar{y}(s) &= P_{21}(s)\bar{w}(s) + P_{22}(s)\bar{u}(s) \\ \bar{u}(s) &= K(s)\bar{y}(s) \end{aligned}$$

$\implies \bar{u}(s) = K(s)\{$

So,

$$\bar{z}(s) = \mathcal{F}_l\big(P(s), K(s)\big)\bar{w}(s)$$

where

$$\mathcal{F}_l\big(P(s), K(s)\big) = P_{11}(s) + P_{12}(s)K(s)\big(I - P_{22}(s)K(s)\big)^{-1}P_{21}(s)$$

We now seek stabilisation controllers $K(s)$ which make $\mathcal{F}_l\big(P(s), K(s)\big)$ "small".

2.5 $\mathcal{H}_2$ optimal control - state-feedback (a special case)

Let the generalised plant P have realization:

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = \begin{bmatrix} C_1 x \\ u \end{bmatrix}$$
$$y = x \qquad \text{(state-feedback)}$$

which we can also write in the more compact form:

$$\left[\begin{array}{c} \dot{x} \\ \hline z \\ y \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline \begin{bmatrix} C_1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ I & 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline w \\ u \end{array}\right].$$

Assumptions:

A, B_2 controllable
A, C_1 observable

Objective: Find $K(s)$ that achieves

$$\min_{K(s)\text{ stabilizing}} \left\| \mathcal{F}_l\big(P(s), K(s)\big) \right\|_2$$

Solution: Recall that when $x(0) = x_0 \neq 0, w(t) = 0,$

$$\|z\|_2^2 = x_0^T X x_0 + \|(u + B_2^T X x)\|_2^2$$

where $X = X^T$ is the stabilising solution to

$$0 = XA + A^T X + C_1^T C_1 - X B_2 B_2^T X \quad \text{(CARE)}$$

(i.e. the unique solution for which $A - B_2 B_2^T X$ is stable)

Consider the situation $x(0^-) = 0$ and $w(t) = e_i\delta(t)$. This case is equivalent to the one corresponding to $x(0^+) = B_1 e_i$ and $w(t) = 0$, for which

$$\left\| z(t)|_{w(t)=e_i\delta(t)} \right\|_2^2 = e_i^T B_1^T X B_1 e_i + \left\| (u + B_2^T X x) \Big|_{x(0^+)=B_1 e_i} \right\|_2^2$$

and $X = X^T$ is the stabilising solution of the same CARE.

Define

$$v(t) = u(t) + B_2^T X x(t)$$

Let $T_{w \to v}$ be the closed loop transfer function from w to v. Then

$$\frac{1}{2\pi} \|T_{w \to v}\|_2^2 = \sum_i \left\| v(t)|_{w(t)=e_i\delta(t)} \right\|_2^2 = \sum_i \left\| (u + B_2^T X x) \Big|_{x(0^+)=B_1 e_i} \right\|_2^2$$

therefore

$$\frac{1}{2\pi} \|\mathcal{F}_l(P(s), K(s))\|_2^2 = \text{trace}\left(B_1^T X B_1\right) + \frac{1}{2\pi} \left\| T_{w \to \underbrace{u + B_2^T X x}} \right\|_2^2$$

$T_{w \to v}$ can be made equal to 0 by choosing

$$\bar{u}(s) = -B_2^T X \bar{x}(s)$$

which corresponds to chosing

$$K(s) = -B_2^T X \text{ (constant feedback gain)}$$

Summary:

$$\min_{K(s) \text{ stabilizing}} \left\| \mathcal{F}_l(P(s), K(s)) \right\|_2 = \sqrt{2\pi} \sqrt{\text{trace}(B_1^T X B_1)}$$

which is achieved by the constant feedback gain $K = -B_2^T X$. Notice that $K(s)$ is stabilising by the properties of CARE.

2.6 $\mathcal{H}_2$ optimal control - output feedback

Consider the generalised plant P with realization:

$$\begin{aligned}
\dot{x} &= Ax + B_1 w_1 + B_2 u \\
z &= \begin{bmatrix} C_1 x \\ u \end{bmatrix} \\
y &= C_2 x + w_2
\end{aligned}$$

or

$$\begin{bmatrix} \dot{x} \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & \begin{bmatrix} B_1 & 0 \end{bmatrix} & B_2 \\ \hline \begin{bmatrix} C_1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C_2 & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w \\ u \end{bmatrix}$$

Assumptions:

A, B_2 controllable
A, C_1 observable } appropriate to the state-feedback / full information problem

A, B_1 controllable
A, C_2 observable } appropriate to the estimation / (dual) problem

Objective: Find $K(s)$ that achieves

$$\min_{K(s) \text{ stabilizing}} \left\| \mathcal{F}_l\big(P(s), K(s)\big) \right\|_2$$

Let $X = X^T$ be the stabilising solution (i.e. the one such that $A - B_2 B_2^T X$ is stable) to

$$0 = XA + A^T X + C_1^T C_1 - X B_2 B_2^T X \quad \text{(CARE)}$$

Then

$$\frac{1}{2\pi} \|\mathcal{F}_l(P(s), K(s))\|_2^2 = \text{trace}\left(B_1^T X B_1\right) + \frac{1}{2\pi} \|T_{w \to \underbrace{u + B_2^T X x}}\|_2^2$$

and

$$T_{w \to v} = \mathcal{F}_l(\tilde{P}, K)$$

where $\tilde{P}$ has realization

$$\begin{bmatrix} \dot{x} \\ \hline v \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & \begin{bmatrix} B_1 & 0 \end{bmatrix} & B_2 \\ \hline F & 0 & I \\ C_2 & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w \\ u \end{bmatrix}$$

and $F = B_2^T X$

Duality:

Note that $\|G(s)\|_2 = \|G(s)^T\|_2$.

If $G(s) = C(sI - A)^{-1} B + D$, then

$$G(s)^T =$$

i.e. $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \implies G(s)^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$

Furthermore,

$$\mathcal{F}_l(\tilde{P}, K)^T = \mathcal{F}_l(\tilde{P}^T, K^T)$$

Using duality,

$$\|\mathcal{F}_l(\tilde{P}, K)\|_2 = \|\mathcal{F}_l(\tilde{P}^T, K^T)\|_2$$

and $\tilde{P}^T$ has the realization

$$\begin{bmatrix} \dot{\tilde{x}} \\ \hline \tilde{v} \\ \tilde{y} \end{bmatrix} = \left[\begin{array}{c|cc} A^T & F^T & C_2^T \\ \hline \begin{bmatrix} B_1^T \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ B_2^T & I & 0 \end{array}\right] \begin{bmatrix} \tilde{x} \\ \hline \tilde{w} \\ \tilde{u} \end{bmatrix}$$

(Note: $\tilde{x}$, $\tilde{v}$, $\tilde{y}$, $\tilde{w}$ and $\tilde{u}$ are fictitious signals, which bear no relation to the original variables.)
We can now apply the state-feedback results to get

$$\frac{1}{2\pi}\left\|\mathcal{F}_l\left(\tilde{P}(s)^T, K(s)^T\right)\right\|_2^2 = \text{trace}\left(FYF^T\right) + \frac{1}{2\pi}\|T_{\tilde{w} \to \tilde{u} + C_2 Y \tilde{x}}\|_2^2$$

where $Y = Y^T$ is the stabilising solution (i.e. the one such that $A - YC_2^T C_2$ is stable) to

$$0 = YA^T + AY + B_1 B_1^T - YC2^T C_2 Y \quad \text{(FARE)}$$

Can we achieve $\tilde{u} = -\underbrace{C_2 Y}_{H^T} \tilde{x}$?

Note that

$$\dot{\tilde{x}} = A^T \tilde{x} + F^T \tilde{w} + C_2^T \tilde{u}$$
$$\tilde{y} = B_2^T \tilde{x} + \tilde{w}$$

So, let

$$\dot{\tilde{x}}_k = A^T \tilde{x}_k + F^T \underbrace{(\tilde{y} - B_2^T \tilde{x}_k)} + C_2^T \tilde{u}$$

Now, if $\tilde{x}_k(t) = \tilde{x}(t)$ then $\dot{\tilde{x}}_k(t) = \dot{\tilde{x}}(t)$
So, if we let $\tilde{x}_k(0^-) = \tilde{x}(0^-) = 0$, then $\tilde{x}_k(t) = \tilde{x}(t)$ for all t. We can then put

Hence, the optimal K^T has realisation

$$\left[\begin{array}{c} \dot{\tilde{x}}_k \\ \hline \tilde{u} \end{array}\right] = \left[\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right] \left[\begin{array}{c} \tilde{x}_k \\ \hline \tilde{y} \end{array}\right]$$

and the optimal K for the original problem has the realisation

$$\left[\begin{array}{c} \dot{x}_k \\ \hline u \end{array}\right] = \left[\begin{array}{c|c} A - B_2 F - H C_2 & -H \\ \hline F & 0 \end{array}\right] \left[\begin{array}{c} x_k \\ \hline y \end{array}\right]$$

where

$F = B_2^T X$

$H = Y C_2^T$

and X and Y are the stabilising solution of (CARE) and (FARE) respectively.

This optimal K achieves

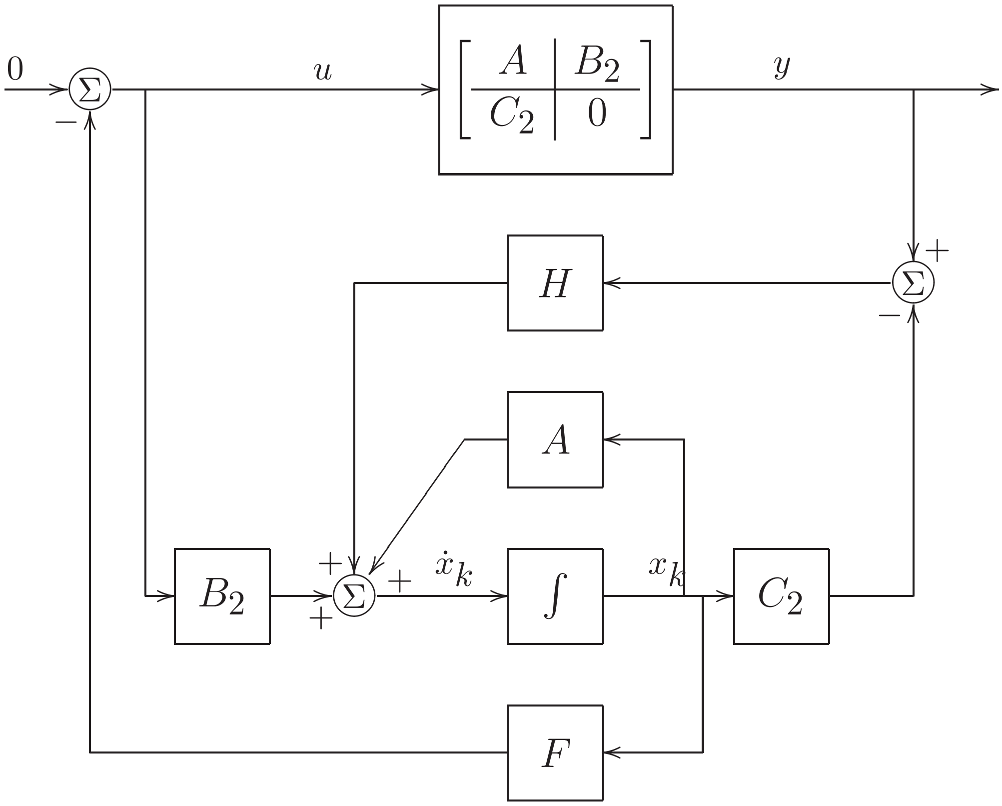
$$\frac{1}{2\pi} \|\mathcal{F}_l(P, K)\|_2^2 = \underbrace{\operatorname{trace}\left(B_1^T X B_1\right)} + \underbrace{\operatorname{trace}\left(F Y F^T\right)}$$

Observer Form:

Another realisation of the optimal K (characterised by the same transfer function) is

$$\begin{bmatrix} \dot{x}_k \\ \hline u \end{bmatrix} = \left[\begin{array}{c|c} A - B_2F - HC_2 & +H \\ \hline -F & 0 \end{array}\right] \begin{bmatrix} x_k \\ \hline y \end{bmatrix}$$

which has an observer form.



$$\text{Closed-loop poles of optimal } K = \lambda_i \underbrace{(A - B_2F)}_{\text{stable}} \cup \lambda_i \underbrace{(A - HC_2)}_{\text{stable}}.$$

So it is a stabilising controller.