Part IIB/EIST Part II, Module 4F2

Robust Multivariable Control

Guy-Bart Stan

HANDOUT 3

$\mathcal{H}_\infty$ optimal control

G. Stan 2009

References

1. M. Green and D. Limebeer, “Linear Robust Control”, Prentice Hall, 1995

2. T. Başar and P. Bernhard, “H^∞-Optimal Control and Related Minimax Design Problems”, Birkhäuser, Second Edition, 1991

3. T. Başar and G. J. Olsder, “Dynamic Non-cooperative Game Theory”, Academic Press, Second Edition, 1995 (SIAM Classics in Applied Mathematics, number 23, 1998)

Contents

3.1 Calculating the $\mathcal{H}_\infty$-norm

Consider the stable linear system $G(s)$ with state-space realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= 0 \end{aligned}$$

where A has all its eigenvalues in the left half plane.

Recall that the $\mathcal{H}_\infty$ norm of G, $\|G\|_\infty$ has two interpretations:

- $\|G\|_\infty = \sup_\omega \bar{\sigma}\big(G(j\omega)\big)$
- $\|G\|_\infty = \sup_{\hat{u}\neq 0} \frac{\|G\hat{u}\|_2}{\|\hat{u}\|_2} = \sup_{u\neq 0} \frac{\|y\|_2}{\|u\|_2}$ where $\hat{y}(s) = G(s)\hat{u}(s)$.

The $\mathcal{H}_\infty$ norm could be estimated numerically using the first interpretation, e.g. by griding over frequency. For proofs it is better to use the second definition.

Assume we want to know whether the $\mathcal{H}_\infty$ norm is smaller than $\gamma > 0$

$$\begin{aligned} \|G\|_\infty \leq \gamma &\iff \|y(t)\|_2 \leq \gamma \|u(t)\|_2 \text{ for all } u \in \mathcal{L}_{2,[0,\infty)} \\ &\iff \|y(t)\|_2^2 - \gamma^2 \|u(t)\|_2^2 \leq 0 \text{ for all } u \in \mathcal{L}_{2,[0,\infty)} \end{aligned}$$

Let $V = x^T X x$ for some $X = X^T > 0$. If we can ensure somehow

$$\frac{dV}{dt} \leq \gamma^2 u^T u - y^T y$$

for all $u \in \mathcal{L}_{2,[0,\infty)}$, then we would have

$$\int_0^\infty \frac{dV}{dt} dt = \underbrace{V(\infty)}_{\to 0} - \underbrace{V(0)}_{=0} \leq \gamma^2 \|u\|_2^2 - \|y\|_2^2 \implies \|y\|_2 \leq \gamma \|u\|_2$$

Now,

$$\begin{aligned} \dot{V} + y^T y - \gamma^2 u^T u &= \\ &= (Ax + Bu)^T X x + x^T X (Ax + Bu) + x^T C^T C x - \gamma^2 u^T u \\ &= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA + C^T C & XB \\ B^T X & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned}$$

So,

$$\max_u \left(\dot{V} + y^T y - \gamma^2 u^T u \right) = x^T \left(A^T X + XA + C^T C + \frac{1}{\gamma^2} X B B^T X \right) x$$

Therefore, if the Riccati equation

$$A^T X + XA + C^T C + \frac{1}{\gamma^2} X B B^T X = 0$$

has a solution $X = X^T > 0$ then

$$\|G\|_\infty \leq \gamma.$$

The condition turns out to be "if and only if" ("only if" proof omitted). This condition is easily checked algebraically. A bisection algorithm can then be used to find the smallest γ for which this Riccati equation has a solution.

3.2 $\mathcal{H}_\infty$ optimal control

Consider the generalised plant P with realization:

$$\begin{aligned} \dot{x} &= Ax + B_1 w_1 + B_2 u \\ z &= \begin{bmatrix} C_1 x \\ u \end{bmatrix} \\ y &= C_2 x + w_2 \\ x(0) &= 0 \end{aligned}$$

or

$$\begin{bmatrix} \dot{x} \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & \begin{bmatrix} B_1 & 0 \end{bmatrix} & B_2 \\ \hline \begin{bmatrix} C_1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C_2 & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w \\ u \end{bmatrix}$$

$$x(0) = 0$$

Assumptions: (same as for $\mathcal{H}_2$ case)

A, B_2 controllable A, C_1 observable	appropriate to the state-feedback / full information problem
A, B_1 controllable A, C_2 observable	appropriate to the estimation / (dual) problem

Objective: Find a stabilising K such that

$$\|\mathcal{F}_l(P(s), K(s))\|_\infty \le \gamma$$

(i.e. $\mathcal{F}_l(P(s), K(s))$ is stable and $\|z\|_2^2 \le \gamma^2 \|w\|_2^2$ for all $w \in \mathcal{L}_{2,[0,\infty)}$.)

Solution: Let $V = x^T X x$, $X = X^T$. If we can ensure somehow that

$$\frac{dV}{dt} \leq \gamma^2 w^T w - z^T z$$

for all $w \in \mathcal{L}_{2,[0,\infty)}$, then we would have

$$\int_0^\infty \frac{dV}{dt} dt = V(\infty) - \overbrace{V(0)}^{=0} \leq \gamma^2 \|w\|_2^2 - \|z\|_2^2.$$

(V is actually the value function for the the continuous time version of the non-cooperative game (worst disturbance) introduced on page 11 of Handout 1 and X solves a modification of the HJB equation known as the Issacs equation - but we don't need to know that here!)

Completing the squares, we can write

$$\frac{dV}{dt} + z^T z - \gamma^2 w^T w =$$

$$= x^T (XA + A^T X + C_1^T C_1 - X B_2 B_2^T X + \gamma^{-2} X B_1 B_1^T X) x +$$

$$(u + B_2^T X x)^T (u + B_2^T X x) - \gamma^2 (w - \frac{1}{\gamma^2} \begin{bmatrix} B_1^T \\ 0 \end{bmatrix} X x)^T (w - \frac{1}{\gamma^2} \begin{bmatrix} B_1^T \\ 0 \end{bmatrix} X x) \quad (*)$$

So, if X is chosen to satisfy

(1) $XA + A^T X + C_1^T C_1 - X(B_2 B_2^T - \gamma^{-2} B_1 B_1^T) X = 0$

(2) $A - B_2 B_2^T X$ stable, closed-loop "A" matrix
when $u = -\boldsymbol{B_2^T X x}$

$w = \mathbf{0}$ (best disturbance)

(as system must be stable when there is no disturbance)

(3) $A - B_2 B_2^T X + \gamma^{-2} B_1 B_1^T X$ stable, closed-loop "A" matrix
when $u = -\boldsymbol{B_2^T X x}$

$w = \boldsymbol{\frac{1}{\gamma^2} \begin{bmatrix} B_1^T \\ 0 \end{bmatrix} X x}$ (worst disturbance)

and **if we were able to choose $u = -B_2^T X x$ (state f/b)** then we would obtain $\gamma^2 \|w\|_2^2 - \|z\|_2^2 \geq 0$

Facts:

i) ($\Rightarrow$) At most one solution to (1) satisfies (3),

ii) ($\Rightarrow$) (2) is then satisfied $\iff$ this solution satisfies $X = X^T > 0$.

iii) ($\Leftarrow$) If there exists a stabilising $K(s)$ such that $\|\mathcal{F}_l(P(s), K(s))\|_\infty < \gamma$, then a solution to (1), (2) and (3) exists.

Choose this X, and integrate $(*)$ from $t = 0$ to $t = \infty$

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 + \overbrace{x(\infty)^T X x(\infty)}^{\to 0} - \overbrace{x(0)^T X x(0)}^{=0} = \left\| u + B_2^T X x \right\|_2^2 - \gamma^2 \left\| w - \frac{1}{\gamma^2}\begin{bmatrix} B_1^T \\ 0 \end{bmatrix} X x \right\|_2^2$$

State feedback case, choose $u = -B_2^T X x$:
LHS $\leq 0 \iff \|T w \to z\|_\infty \leq \gamma$

Partial info case:

$$\text{RHS} \leq 0 \iff \left\| T_{\left\{ w - \frac{1}{\gamma^2}\begin{bmatrix} B_1^T \\ 0 \end{bmatrix} X\, x \right\} \to \left\{ u + B_2^T X x \right\}} \right\|_\infty \leq \gamma$$

So,

$$\|\mathcal{F}_l(P, K)\|_\infty \leq \gamma \iff \left\| T_{\left\{ w - \frac{1}{\gamma^2}\begin{bmatrix} B_1^T \\ 0 \end{bmatrix} X x \right\} \to \left\{ u + B_2^T X x \right\}} \right\|_\infty \leq \gamma$$

Furthermore

$$\underbrace{{}^{T}\left\{w - \frac{1}{\gamma^2}\begin{bmatrix} B_1^T \\ 0 \end{bmatrix} Xx\right\}}_{r} \rightarrow \underbrace{\left\{u + B_2^T Xx\right\}}_{v} = \mathcal{F}_l(\tilde{P}, K)$$

where $\tilde{P}$ has the realization

$$\dot{x} = Ax + \begin{bmatrix} B_1 & 0 \end{bmatrix} \overbrace{\left(r + \frac{1}{\gamma^2}\begin{bmatrix} B_1^T \\ 0 \end{bmatrix} Xx\right)}^{w} + B_2 u$$

$$v = u + B_2^T Xx$$

$$y = C_2 x + \begin{bmatrix} 0 & I \end{bmatrix} \overbrace{\left(r + \frac{1}{\gamma^2}\begin{bmatrix} B_1^T \\ 0 \end{bmatrix} Xx\right)}^{w} = C_2 x + \begin{bmatrix} 0 & I \end{bmatrix} r$$

or

$$\begin{bmatrix} \dot{x} \\ \hline v \\ y \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \begin{bmatrix} B_1 & 0 \end{bmatrix} & B_2 \\ \hline F & 0 & I \\ C_2 & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right] \begin{bmatrix} x \\ \hline r \\ u \end{bmatrix}$$

where

$$\hat{A} = A + \frac{1}{\gamma^2} B_1 B_1^T X, \qquad F = B_2^T X$$

So,

$$\|\mathcal{F}_l(P, K)\|_\infty \leq \gamma \iff \left\| {}^{T}\left\{w - \frac{1}{\gamma^2}\begin{bmatrix} B_1^T \\ 0 \end{bmatrix} Xx\right\} \rightarrow \left\{u + B_2^T Xx\right\} \right\|_\infty \leq \gamma$$

$$\iff \left\|\mathcal{F}_l(\tilde{P}, K)\right\|_\infty \leq \gamma \iff \left\|\mathcal{F}_l(\tilde{P}^T, K^T)\right\|_\infty \leq \gamma$$

Invoking duality.

$\tilde{P}^T$ has the realization

$$\left[\begin{array}{c}\dot{\tilde{x}}\\ \hline \tilde{v}\\ \tilde{y}\end{array}\right] = \left[\begin{array}{c|cc}\hat{A}^T & F^T & C_2^T\\ \hline \begin{bmatrix}B_1^T\\ 0\end{bmatrix} & 0 & \begin{bmatrix}0\\ I\end{bmatrix}\\ B_2^T & I & 0\end{array}\right]\left[\begin{array}{c}\tilde{x}\\ \hline \tilde{r}\\ \tilde{u}\end{array}\right]$$

Can now use the state-feedback results to show that, if there exists a $Y = Y^T > 0$ which is the stabilising solution to

$$Y\hat{A}^T + \hat{A}Y + B_1B_1^T - Y\left(C_2^TC_2 - \gamma^{-2}F^TF\right)Y = 0, \quad F = B_2^TX$$

(note that, contrary to the $\mathcal{H}_2$ case, Y depends on X)
then

$$\|\tilde{v}\|_2^2 - \gamma^2\|\tilde{r}\|_2^2 = \|\tilde{u} + C_2Y\tilde{x}\|_2^2 - \gamma^2\|\tilde{r} - \gamma^{-2}FY\tilde{x}\|_2^2$$

Furthermore (as in the $\mathcal{H}_2$ case, p 15-16 of Handout 2) we can achieve

$$\tilde{u} = -\underbrace{C_2Y}_{H^T}\tilde{x}$$

with the controller K^T with realization

$$\left[\begin{array}{c}\dot{\tilde{x}}_k\\ \hline \tilde{u}\end{array}\right] = \left[\begin{array}{c|c}\hat{A}^T - F^TB_2^T - C_2^TH^T & F^T\\ \hline -H^T & 0\end{array}\right]\left[\begin{array}{c}\tilde{x}_k\\ \hline \tilde{y}\end{array}\right]$$

So, one suitable K has the realization

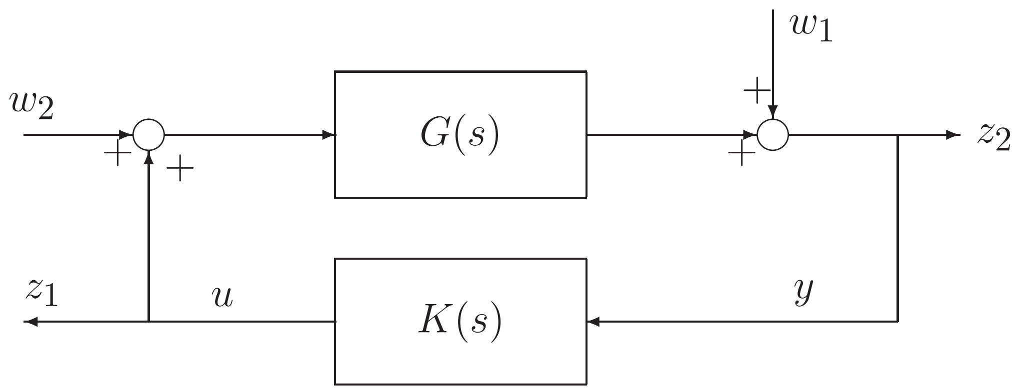
$$\left[\begin{array}{c}\dot{x}_k\\ \hline u\end{array}\right] = \left[\begin{array}{c|c}\hat{A} - B_2F - HC_2 & -H\\ \hline F & 0\end{array}\right]\left[\begin{array}{c}x_k\\ \hline y\end{array}\right]$$

and achieves $\|\mathcal{F}_l(P,K)\|_\infty \le \gamma$.

(Glover et al, 1989)

3.3 The $\mathcal{H}_\infty$ loop-shaping controller

Consider the problem



(see handout 2, LFT)

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ I & G & G \\ I & G & G \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}$$

So,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathcal{F}_l(P, K) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where

$$P = \begin{bmatrix} 0 & 0 & I \\ I & G & G \\ I & G & G \end{bmatrix}$$

(giving

$$\mathcal{F}_l(P, K) = \begin{bmatrix} 0 & 0 \\ I & G \end{bmatrix} + \begin{bmatrix} I \\ G \end{bmatrix} K(I - GK)^{-1} \begin{bmatrix} I & G \end{bmatrix}$$

$$= \begin{bmatrix} K(I - GK)^{-1} & K(I - GK)^{-1}G \\ (I - GK)^{-1} & (I - GK)^{-1}G \end{bmatrix} \quad)$$

If G has the state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ then P has the state-space realization

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & \begin{bmatrix} 0 & B \end{bmatrix} & B \\ \hline \begin{bmatrix} 0 \\ C \end{bmatrix} & \begin{matrix} 0 & 0 \\ I & 0 \end{matrix} & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ C & \begin{bmatrix} I & 0 \end{bmatrix} & 0 \end{array}\right] \begin{bmatrix} x \\ w_1 \\ w_2 \\ u \end{bmatrix}$$

This does not quite meet the assumptions ($D_{11} \neq 0$). However, define

$$\tilde{P} := \left[\begin{array}{c|cc} A & \begin{bmatrix} 0 & \frac{1}{\beta}B \end{bmatrix} & \frac{1}{\beta}B \\ \hline \begin{bmatrix} 0 \\ C \end{bmatrix} & 0 & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ C & \begin{bmatrix} I & 0 \end{bmatrix} & 0 \end{array}\right]$$

where $\beta = \sqrt{1 - 1/\gamma^2}$. Then it can be shown that

$$\|\mathcal{F}_l(P, K)\|_\infty < \gamma \iff \|\mathcal{F}_l(\tilde{P}, \tilde{K})\|_\infty < \gamma$$

where $\tilde{K} = \frac{1}{\beta}K$. (*this is known as a loop-shifting transformation*)
$X = X^T$ must now be the stabilising solution to

$$XA + A^TX + C_1^TC_1 - X(B_2B_2^T - B_1B_1^T/\gamma^2)X = 0$$

i.e. the stabilising solution ($A - BB^TX$ stable) to

$$XA + A^TX + C^TC - XBB^TX = 0 \qquad \text{(CARE)}$$

and $Y = Y^T$ must then be the stabilising solution to

$$Y\hat{A}^T + \hat{A}Y + B_1B_1^T - Y\left(C_2^TC_2 - F^TF/\gamma^2\right)Y = 0, \quad F = B_2^TX \quad (3.1)$$

Finally, it can be shown that if $Z = Z^T$ is the stabilising solution ($A - ZC^TC$ stable) to

$$ZA^T + AZ + BB^T - ZC^TCZ = 0 \qquad \text{(FARE)}$$

and if $\gamma > \sqrt{1 + \lambda_{max}(XZ)}$ then there exists a suitable (i.e. stabilising) Y satisfying (3.1). (actually if and only if)

Conclusion: There exists a controller satisfying $b(G, K) = \frac{1}{\|\mathcal{F}_l(P,K)\|_\infty} > 1/\gamma$ if and only if $\gamma > \sqrt{1 + \lambda_{max}(XZ)}$ where X and Z are the stabilising solutions to (CARE) and (FARE).